Robotics

Robot Localization – Sensor Models and Bayesian Filtering

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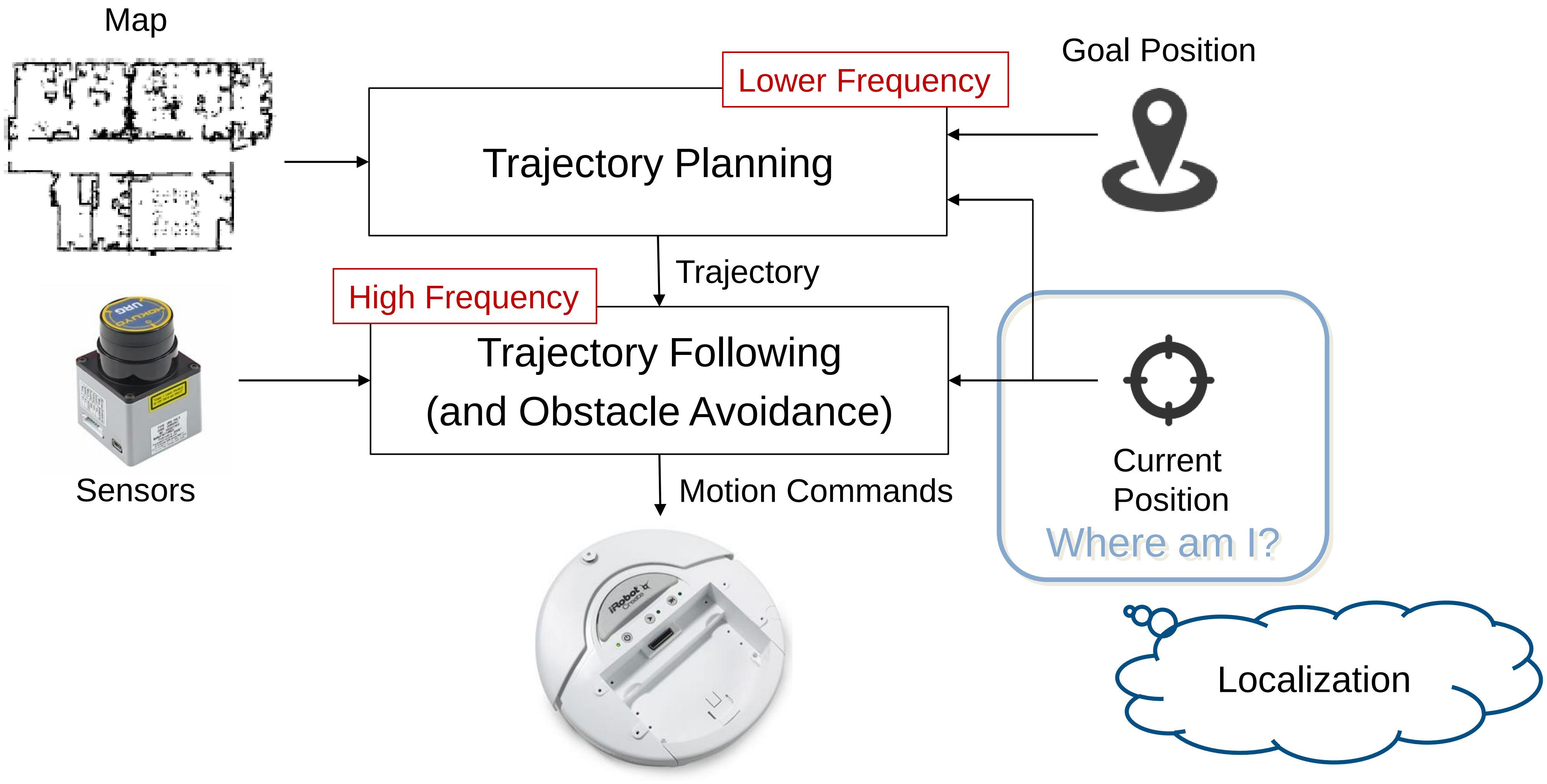
A Simplified Sense-Plan-Act Architecture



Map

Lower Frequency

Goal Position

Trajectory Planning

Trajectory

High Frequency

Trajectory Following
(and Obstacle Avoidance)

Sensors

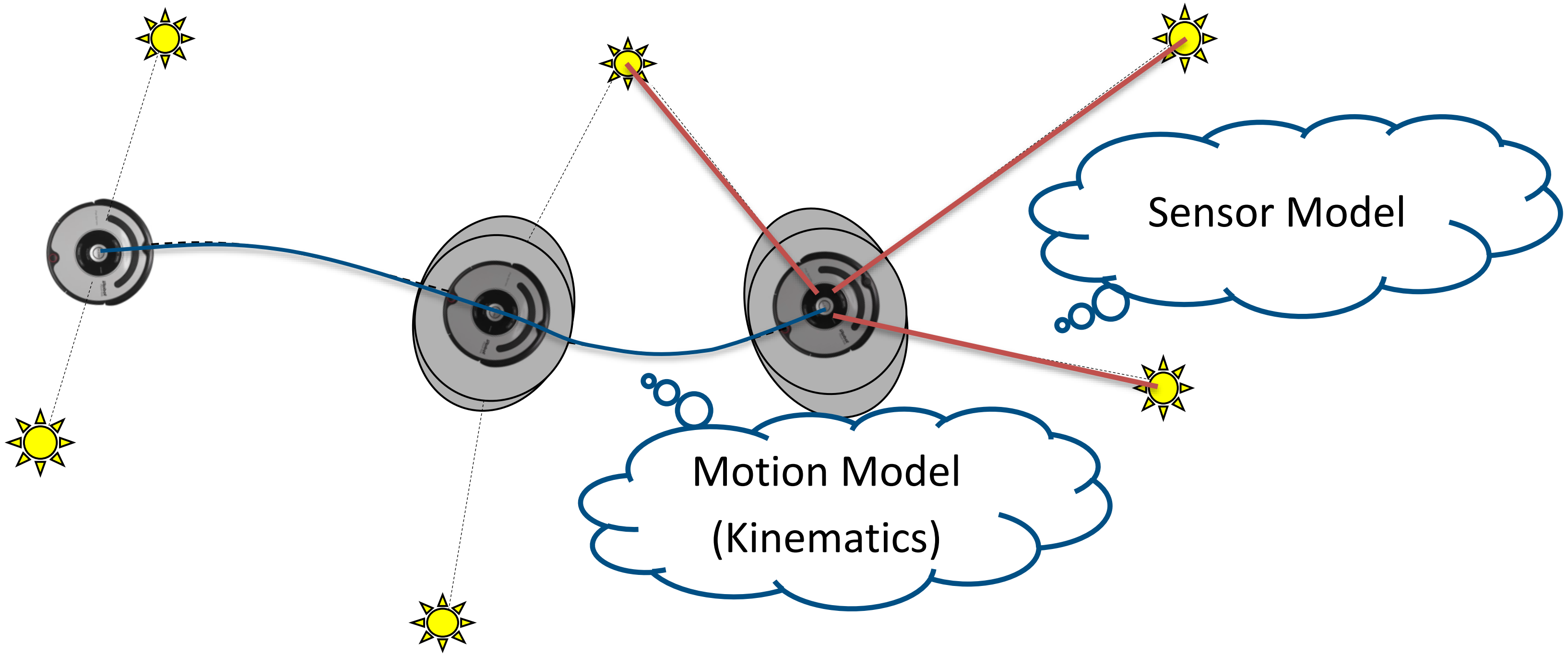
Motion Commands

Current
Position

Where am I?

Localization

Localization with Knowm Map

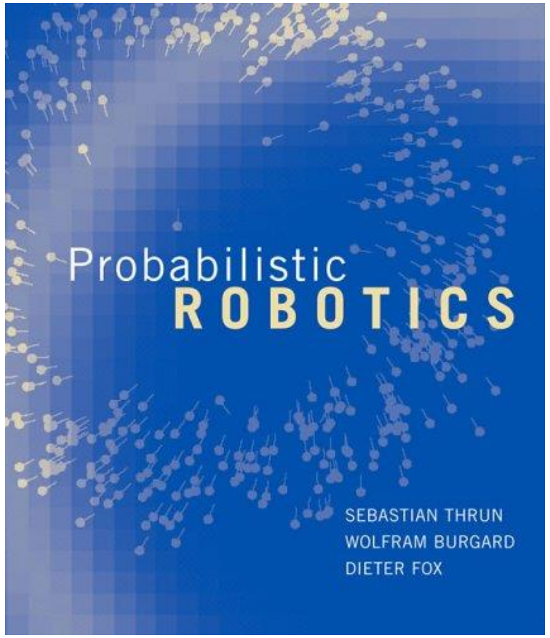


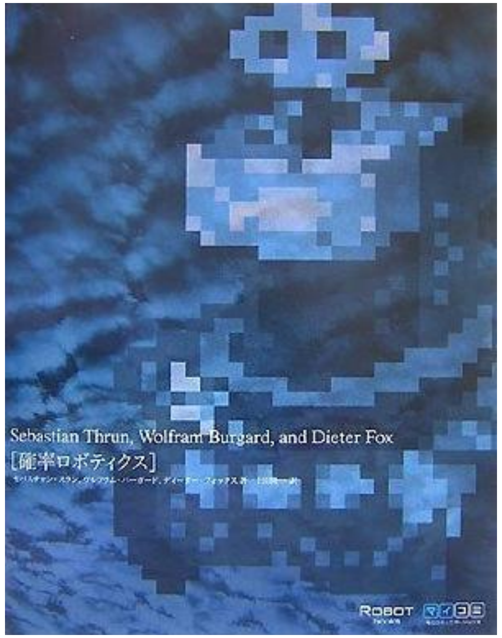
Sensor Model

Motion Model
(Kinematics)

Disclaimer …

Slides from now on have been heavily “inspired” by the teaching material kindly provided with: S. Thrun, D. Fox, W. Burgard. “*Probabilistic Robotics*”*.* MIT Press, 2005


Probabilistic
ROBOTICS
SEBASTIAN THRUN
WOLFRAM BURGARD
DIETER FOX

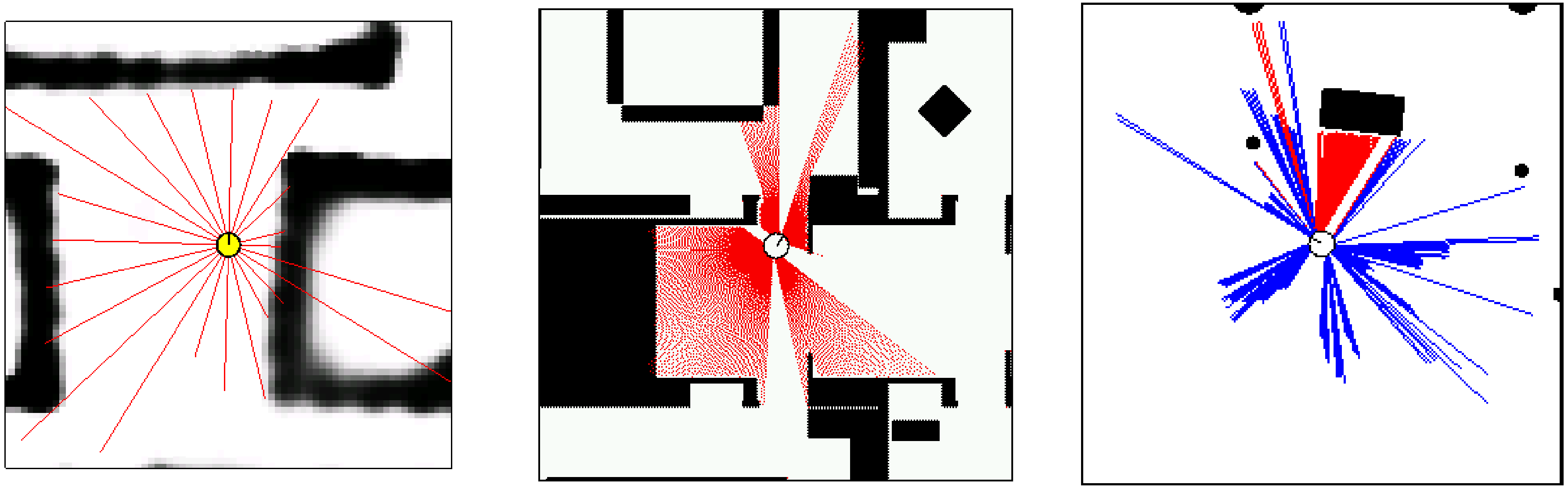

Sebastian Thrun, Wolfram Burgard, and Dieter Fox
[確率ロボティクス]
ROBOT

http://robots.stanford.edu/probabilistic-robotics/

You can refer to the original source for deeper analysis and references on the topic …

Range Sensors Models

The sensor model describes P(z|x), i.e., the probability of a measurement z given that the robot is at position x.



Range Sensors

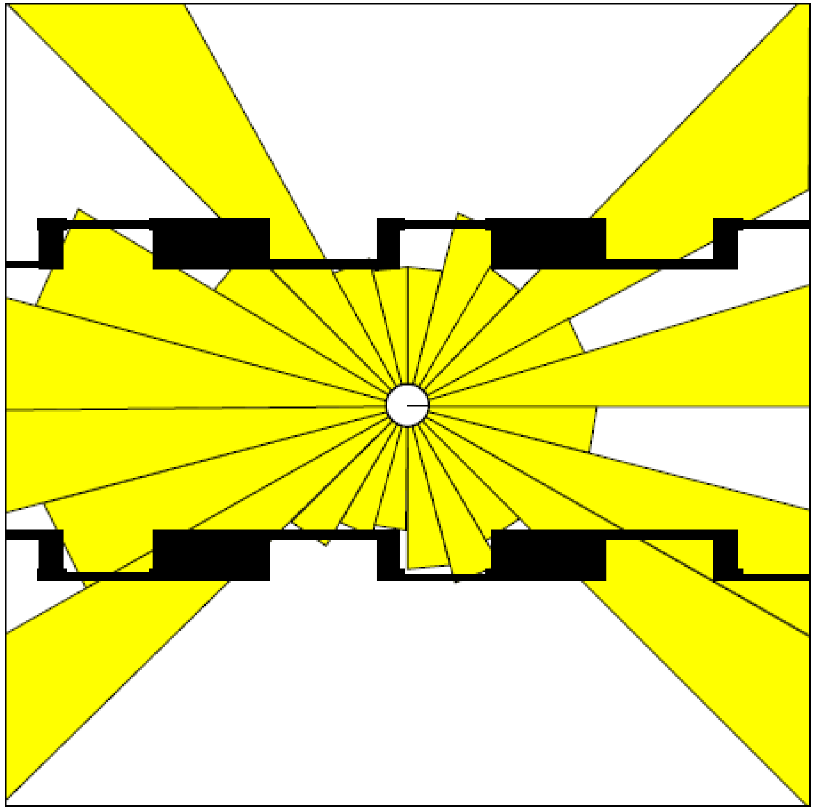
The sensor model describes P(z|x), i.e., the probability of a measurement z given that the robot is at position x.

In particular a scan *z* consists of *K* measurements.

$$z = \{z_1, z_2, \ldots, z_K\}$$

Individual measurements are independent given robot position and surrounding map.

$$P(z \mid x, m) = \prod_{k=1}^{K} P(z_k \mid x, m)$$

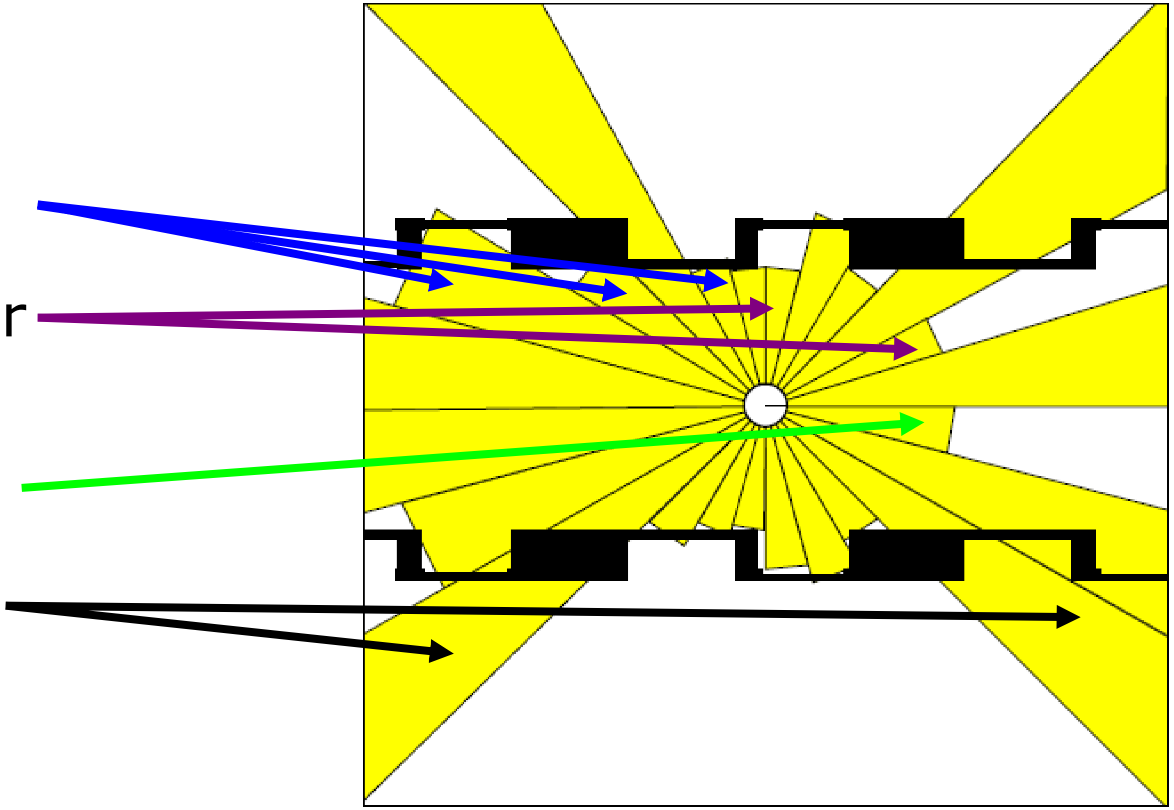


Typical Measurement Errors of an Range Measurements

The sensor model describes P(z|x), i.e., the probability of a measurement z given that the robot is at position x.

Masurements can come from:

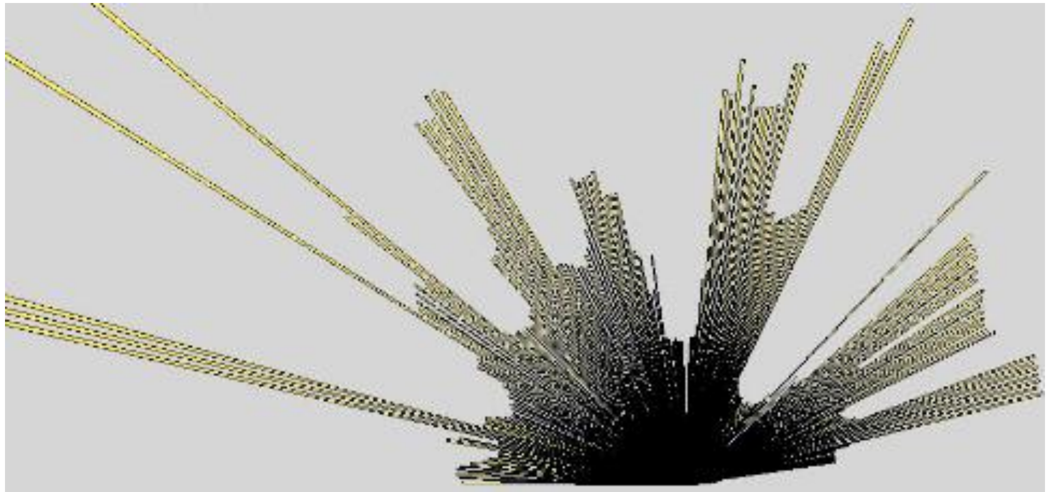
1. Beams reflected by obstacles
2. Beams reflected by persons or caused by crosstalk
3. Random measurements
4. Max range measurements



Distance perception: Laser Range Finder

Lasers are definitely more accurate sensors

- 180 ranges over 180° (up to 360 °)
- 1 to 64 planes scanned, 10-75 scans/s
- <1cm range resolution
- Max range up to 50-80 m
- Issues with mirrors, glass, and matte black.



SICK
HOKUYO
URG

< 1000 €

~ 6000 €

~ 10.000 €

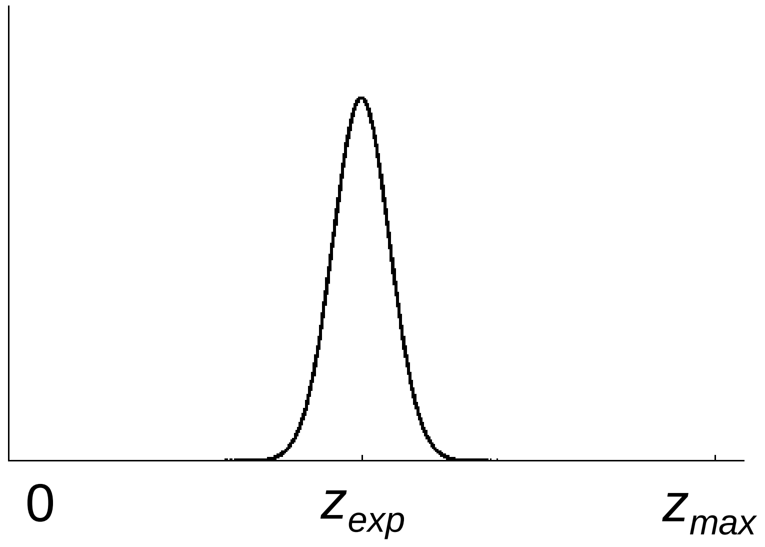
~ 40.000 €

> 80.000 €

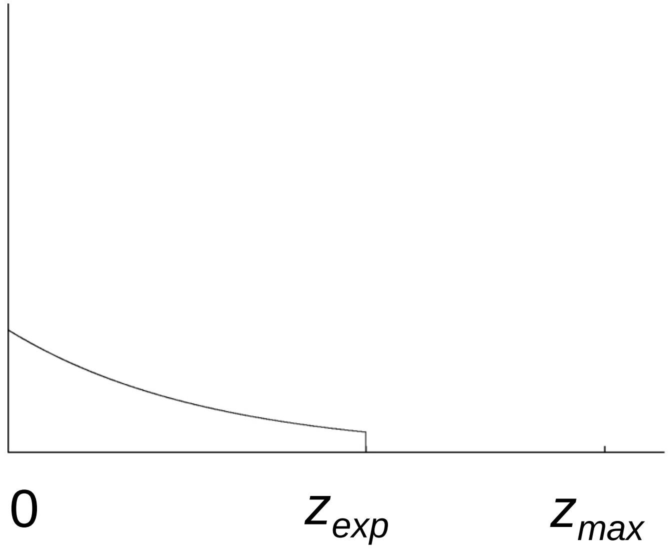
Beam Sensor Model (I)

The laser range finder model describes each single measurement using

Measurement noise



Unexpected obstacles



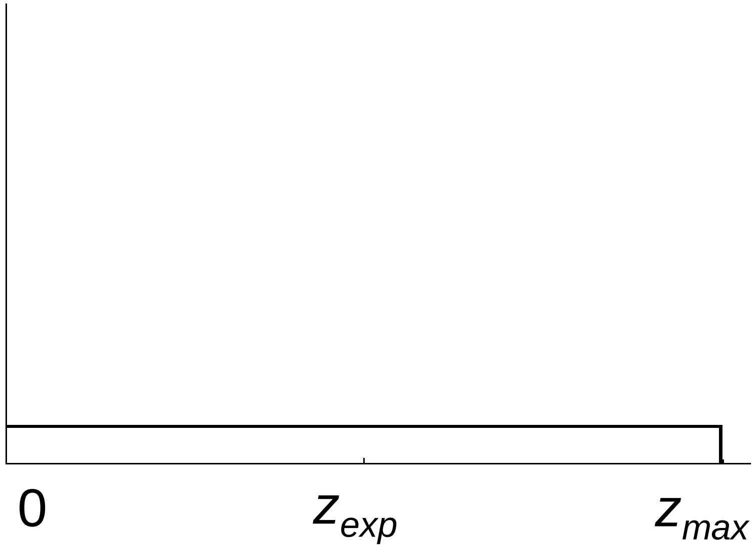
$$P_{hit}(z \mid x, m) = \eta \frac{1}{\sqrt{2\pi b}} e^{-\frac{1}{2}\frac{(z-z_{\exp})^2}{b}}$$

$$P_{\text{unexp}}(z \mid x, m) = \left\{ \begin{array}{cc} \eta\,\lambda\,\mathrm{e}^{-\lambda z} & z < z_{\exp} \\ 0 & otherwise \end{array} \right\}$$

Beam Sensor Model (II)

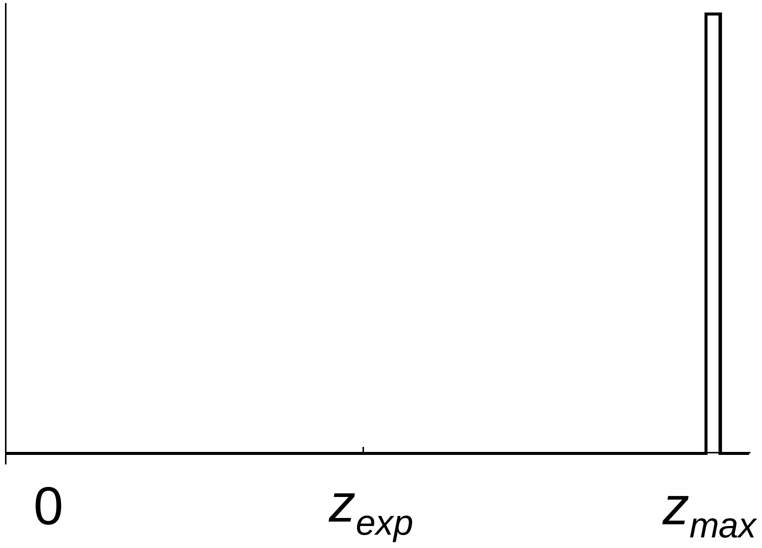
The laser range finder model describes each single measurement using

Random measurement



$$P_{rand}(z \mid x, m) = \eta \frac{1}{z_{\max}}$$

Max range

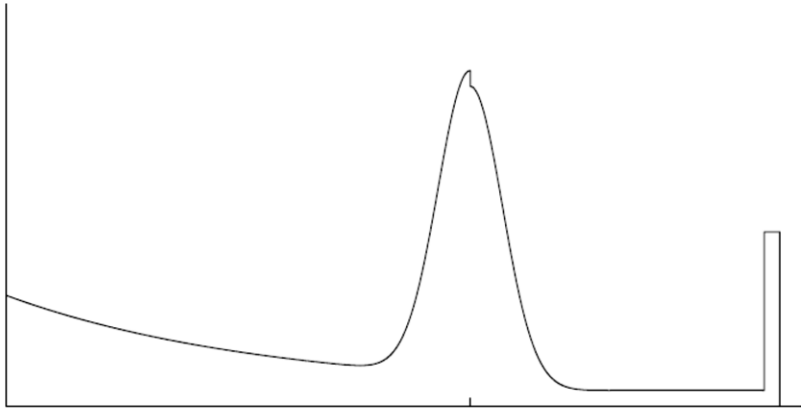


$$P_{\max}(z \mid x, m) = \eta \frac{1}{z_{small}}$$

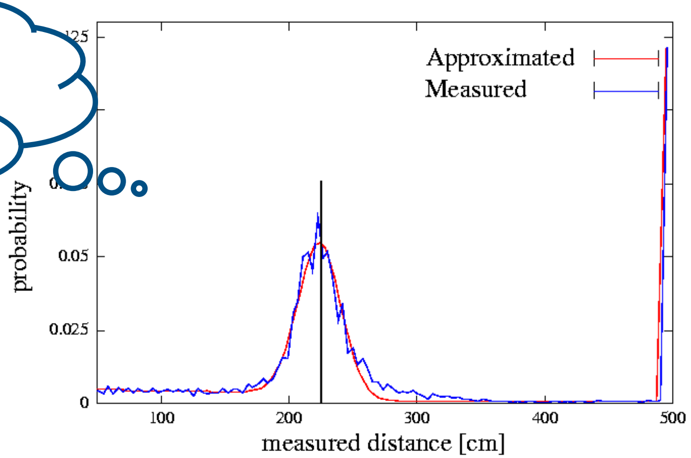
Beam Sensor Model (III)

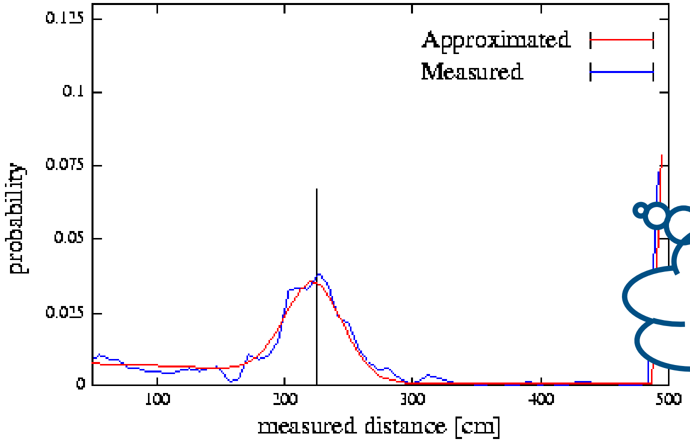
The laser range finder model describes each single measurement using

$$P(z \mid x,m) = \begin{pmatrix} \alpha_{\text{hit}} \\ \alpha_{\text{unexp}} \\ \alpha_{\max} \\ \alpha_{\text{rand}} \end{pmatrix}^T \cdot \begin{pmatrix} P_{\text{hit}}(z \mid x,m) \\ P_{\text{unexp}}(z \mid x,m) \\ P_{\max}(z \mid x,m) \\ P_{\text{rand}}(z \mid x,m) \end{pmatrix}$$



Laser

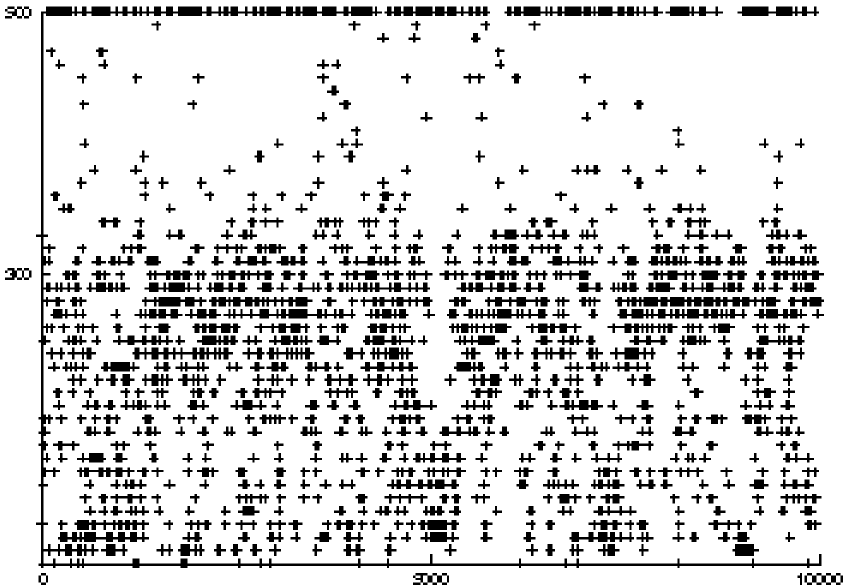




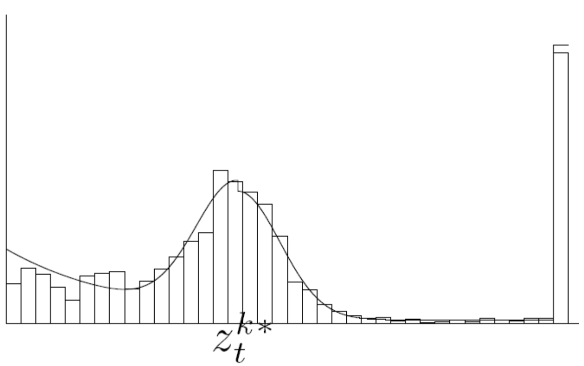
Sonar

Sensor Model Calibration (Sonar)

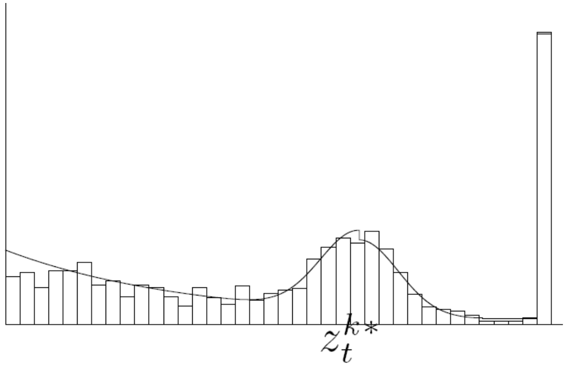
Acquire some data from the sensor, e.g., when the target is at 300 cm and 400 cm



Sonar



300 cm

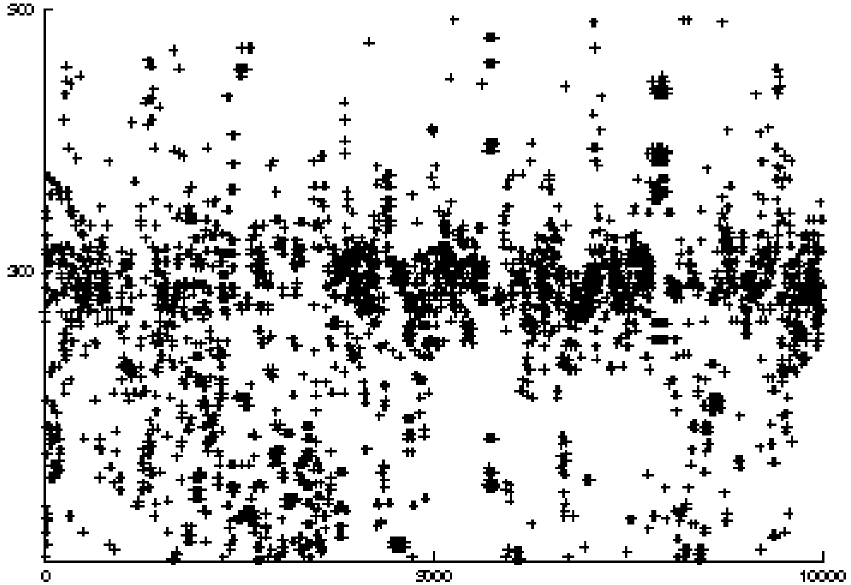


400 cm

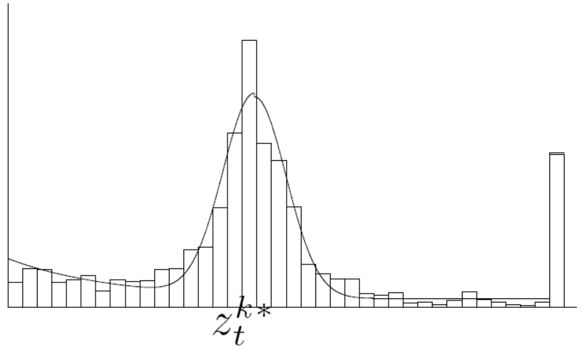
Then estimate the model parameters via maximum likelihood: $P(z \mid z_{\text{exp}})$

Sensor Model Calibration (Laser)

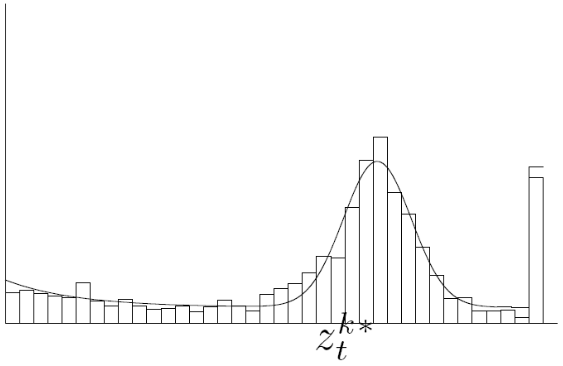
Acquire some data from the sensor, e.g., when the target is at 300 cm and 400 cm



Laser



300 cm

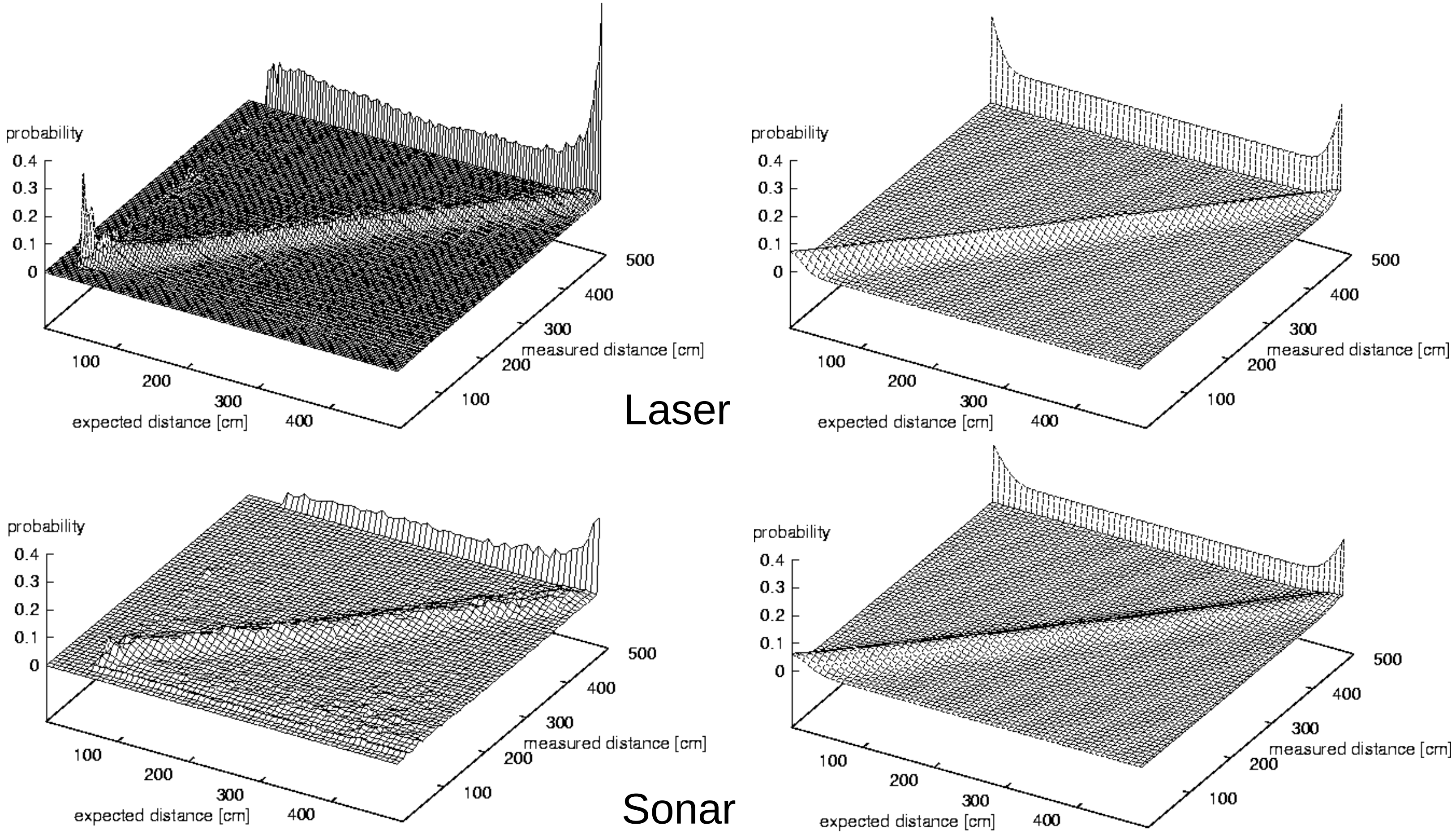


400 cm

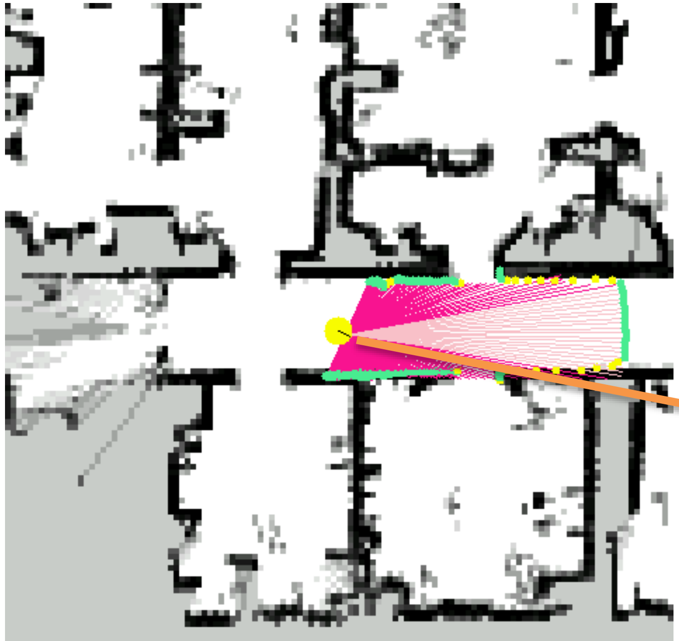
Then estimate the model parameters via maximum likelihood: $P(z \mid z_{\exp})$

Discete Model for Range Sensor

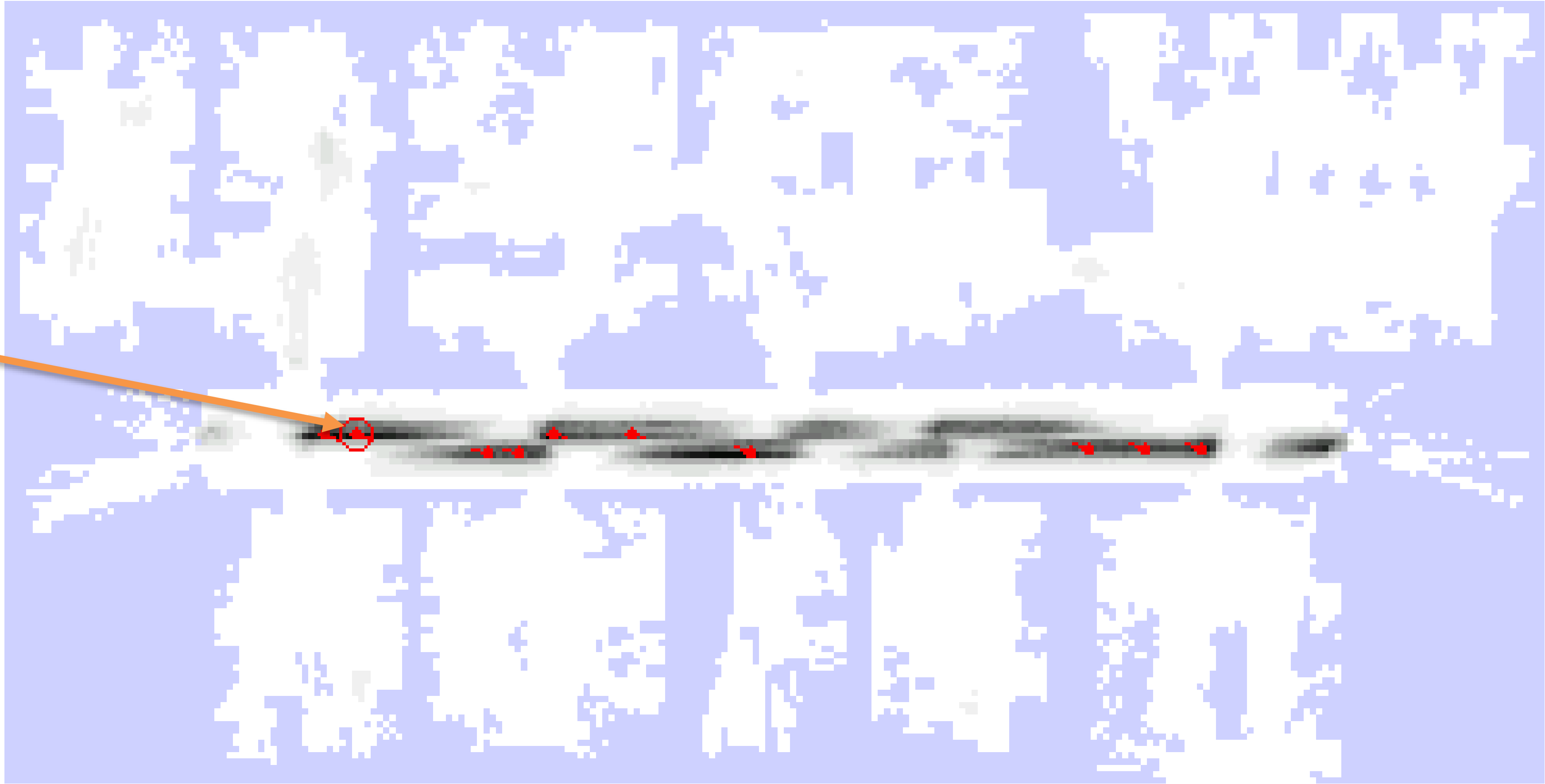
Instead of densities, consider discrete steps along the sensor beam



Sensor Model Likelihood



z



$P(z|x,m)$

Scan Sensor Model

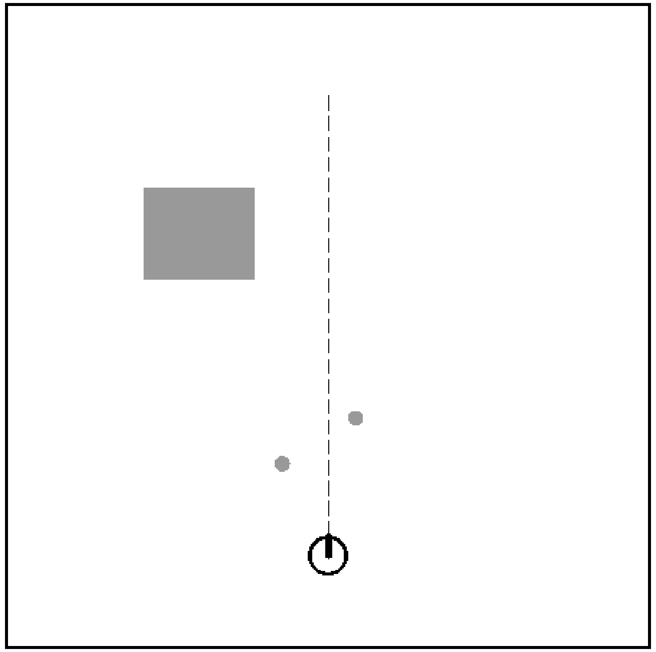
The Beam sensor model assumes independence between beams and between physical causes of measurements and turns out to have some issues:

- Overconfident because of independency assumptions
- Need to learn parameters from data
- A different model should be learned for different angles w.r.t. obstacles
- Inefficient because it uses ray tracing

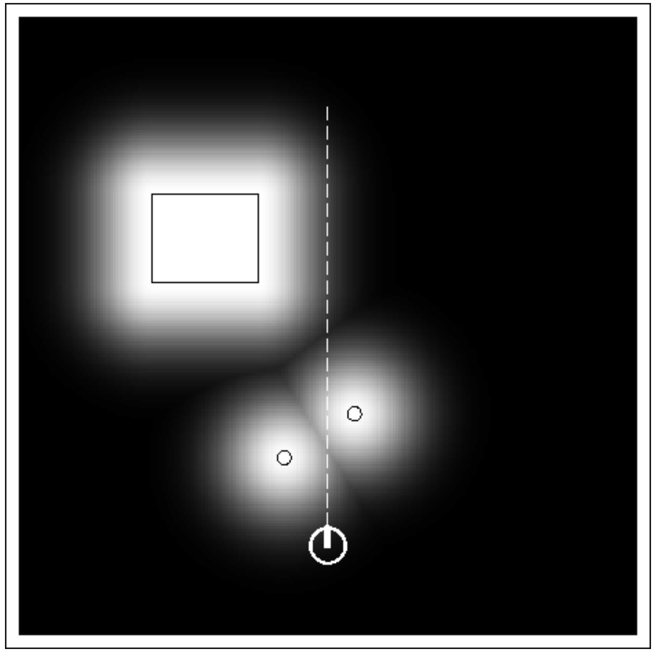
The Scan Sensor Model simplifies Beam Sensor Model with:

- Gaussian distribution with mean at distance to **closest** obstacle,
- Uniform distribution for random measurements, and
- Small uniform distribution for max range measurements

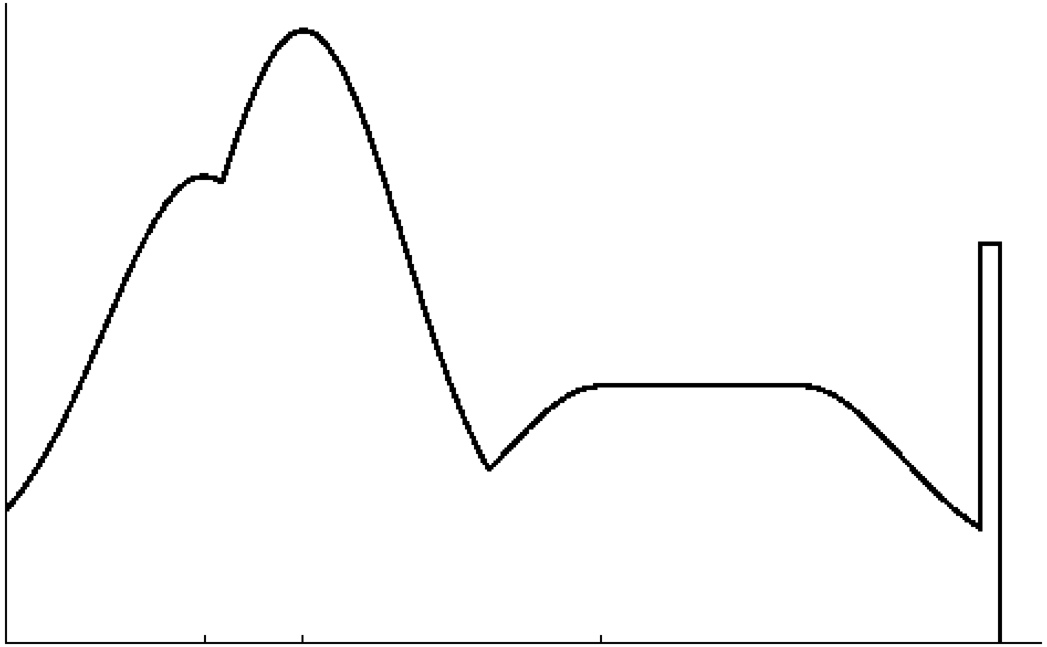
Scan Sensor Model Example



Map *m*

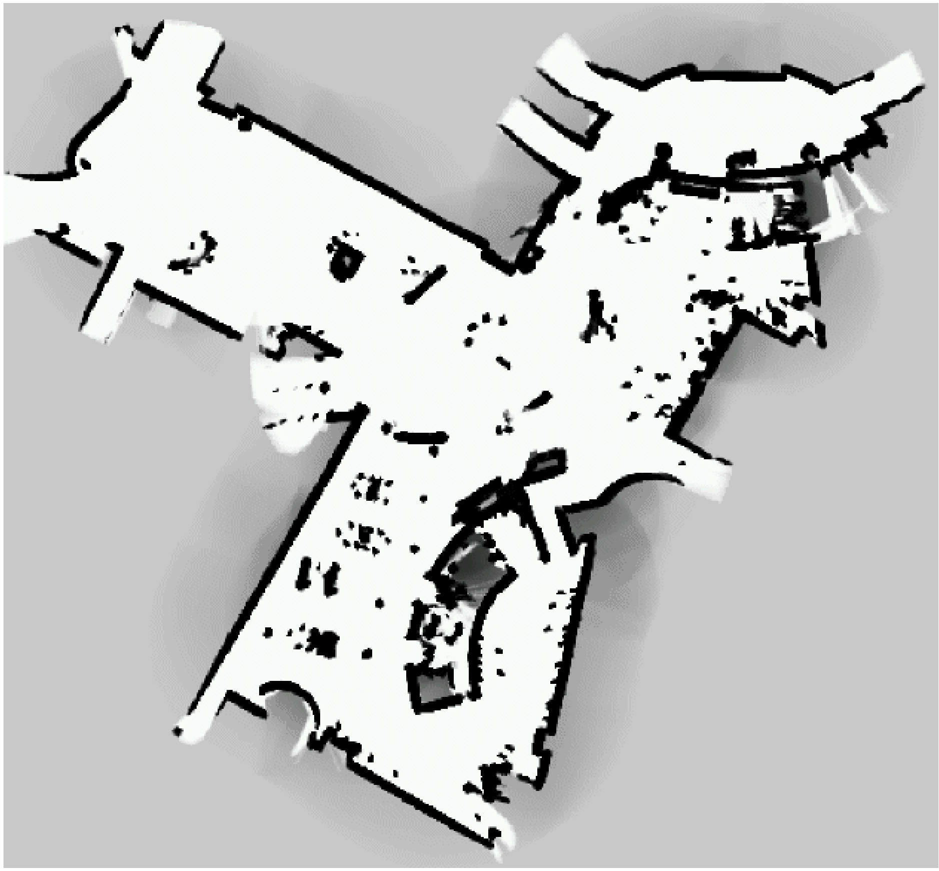


Likelihood field

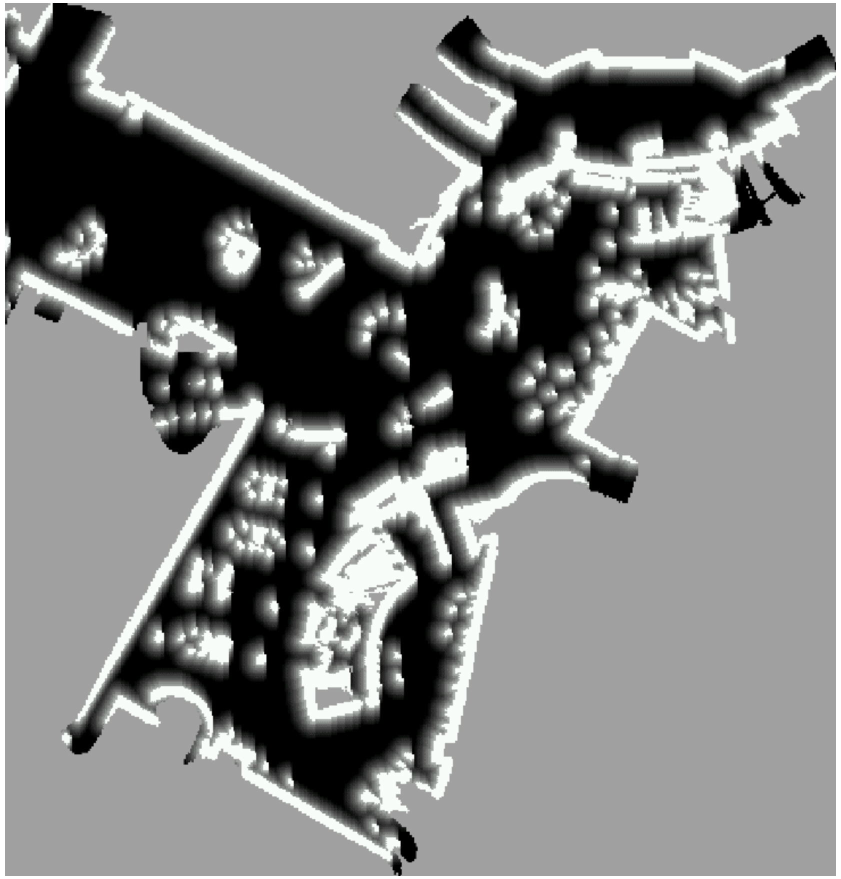


$P(z|x,m)$

San Jose Tech Museum



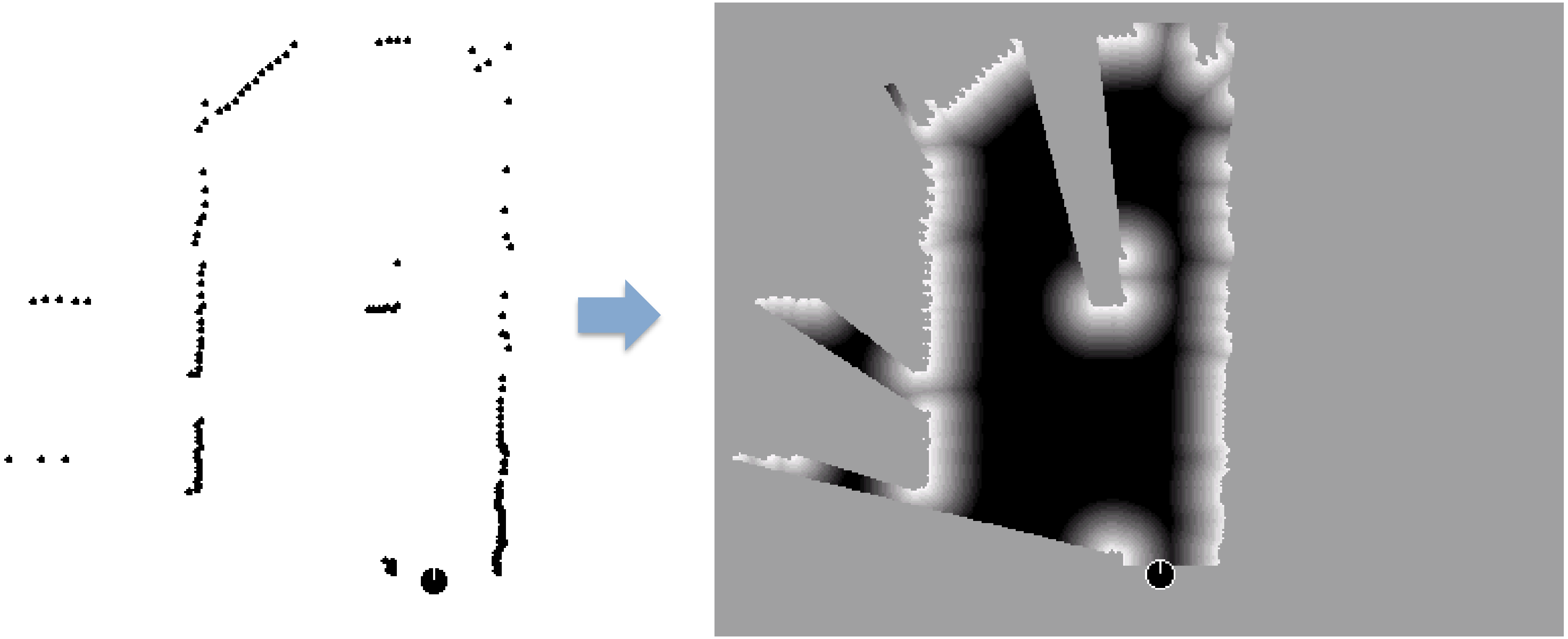
Occupancy grid map



Likelihood field

Scan Matching via Likelihood Field

Extract likelihood field from scan and use it to match different scan:



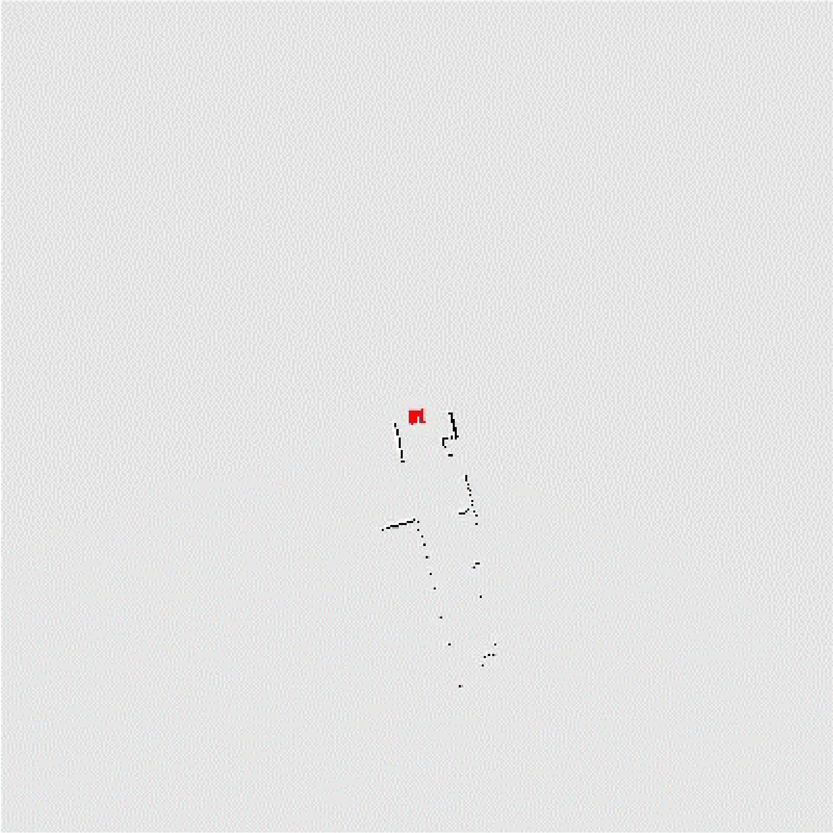
Scan Matching via Likelihood Field

However it does not work with sonars …

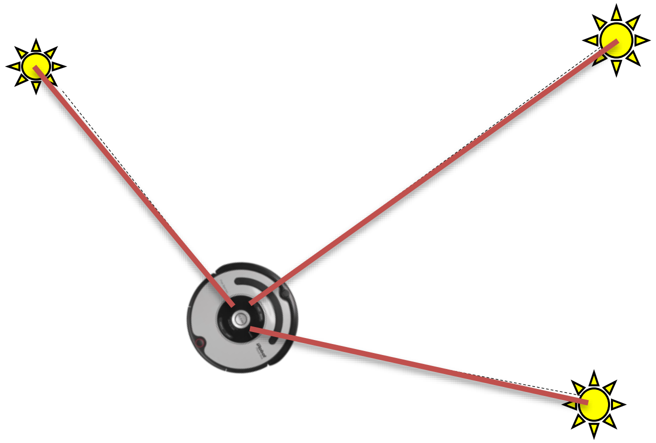
Extract likelihood field from scan and use it to match different scan:

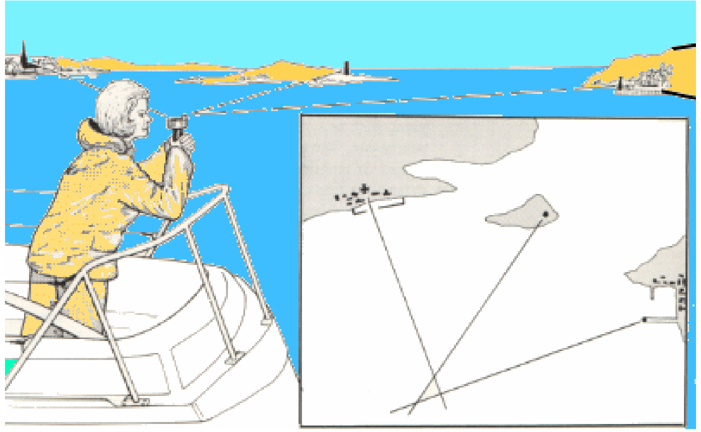
- Highly efficient, uses 2D tables only.
- Smooth with respect to small changes in robot position
- Allows gradient descent pose optimization
- Ignores physical properties of beams.

In this video we are doing more than simple scan matching …



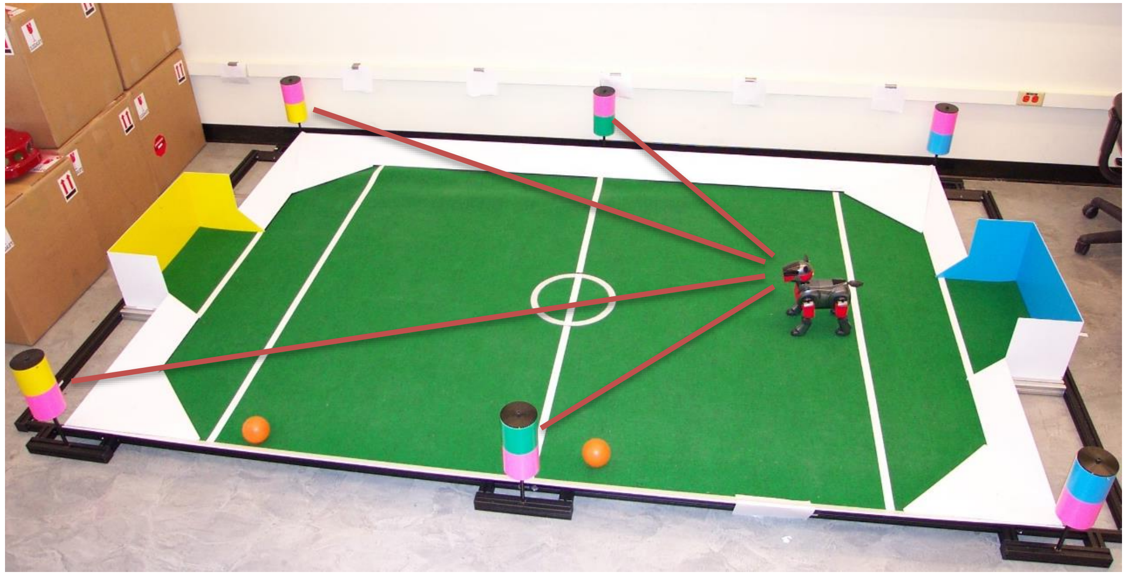
Landmarks





Landmark sensors provides

- Distance (or)
- Bearing (or)
- Distance and bearing



Can be obtained via

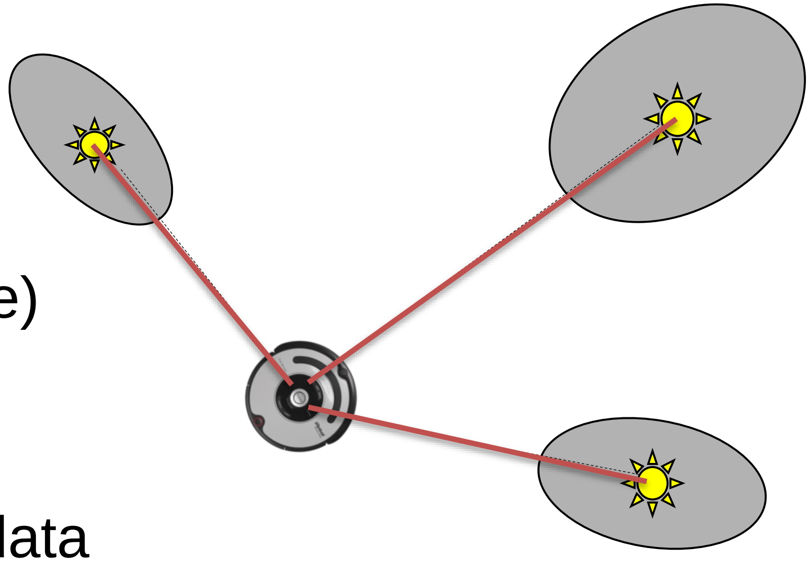
- Active beacons (e.g., radio, GPS)
- Passive (e.g., visual, retro-reflective)

Standard approach is triangulation

Landmark Models with Uncertainty

Explicitly modeling uncertainty in sensing is key to robustness:

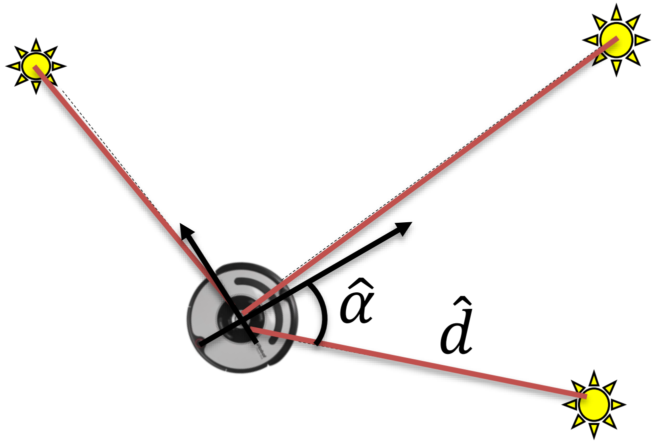
- Determine parametric model for noise free measurement
- Analyze sources of noise (e.g., distance and angle)
- Add adequate noise to parameters (eventually mix in densities for noise)
- Learn (and verify) parameters by fitting model to data



The likelihood of measurement is given by “probabilistically comparing” actual measurements against the expected ones.

Landmark Detection Model

For landmak i in map m, i.e., $m(i)$, the measurement $z = (i, d, \alpha)$ for a robot at position (x, y, θ) is given by



$$\hat{d} = \sqrt{(m_x(i) - x)^2 + (m_y(i) - y)^2}$$

$$\hat{a} = \mathrm{atan2}(m_y(i) - y, m_x(i) - x) - \theta$$

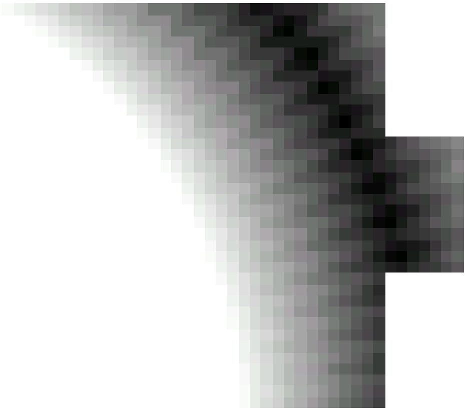
Detection probability might depend on the distance/bearing

$$p_{\mathrm{det}} = \mathrm{prob}(\hat{d} - d, \varepsilon_d) \cdot \mathrm{prob}(\hat{\alpha} - \alpha, \varepsilon_\alpha)$$

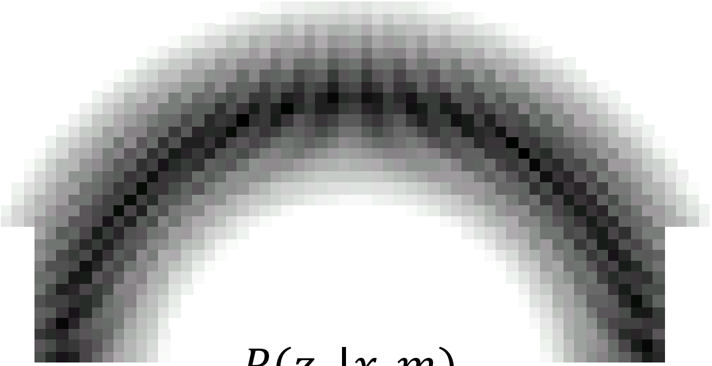
Then we have to take into account false positives too

$$z_{\mathrm{det}} p_{\mathrm{det}} + z_{\mathrm{fp}} P_{\mathrm{uniform}}(z \mid x, m)$$

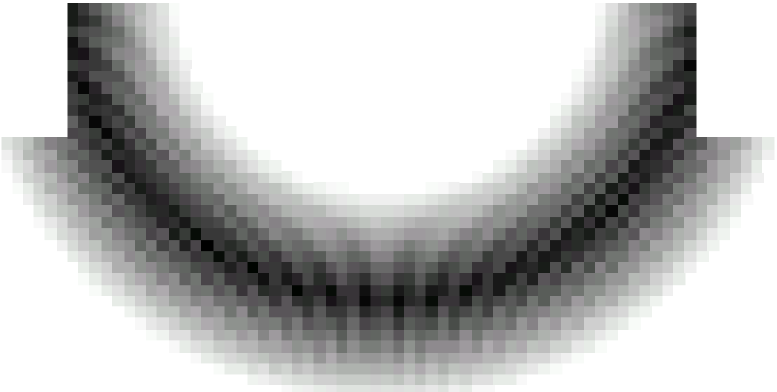
RoboCup Example



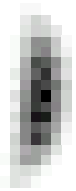
$P(z_1|x,m)$



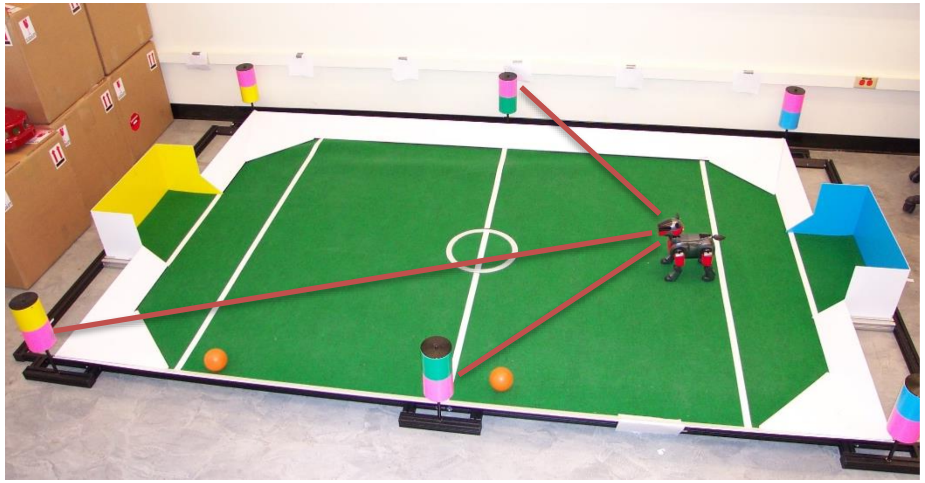
$P(z_2|x,m)$

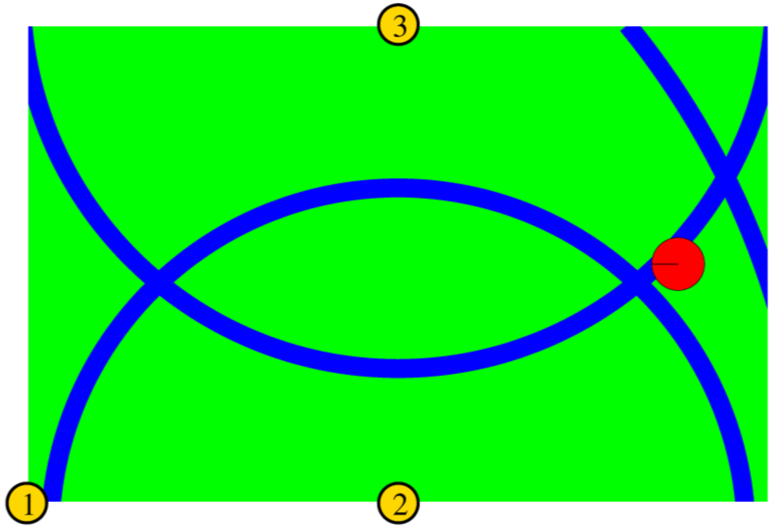


$P(z_3|x,m)$

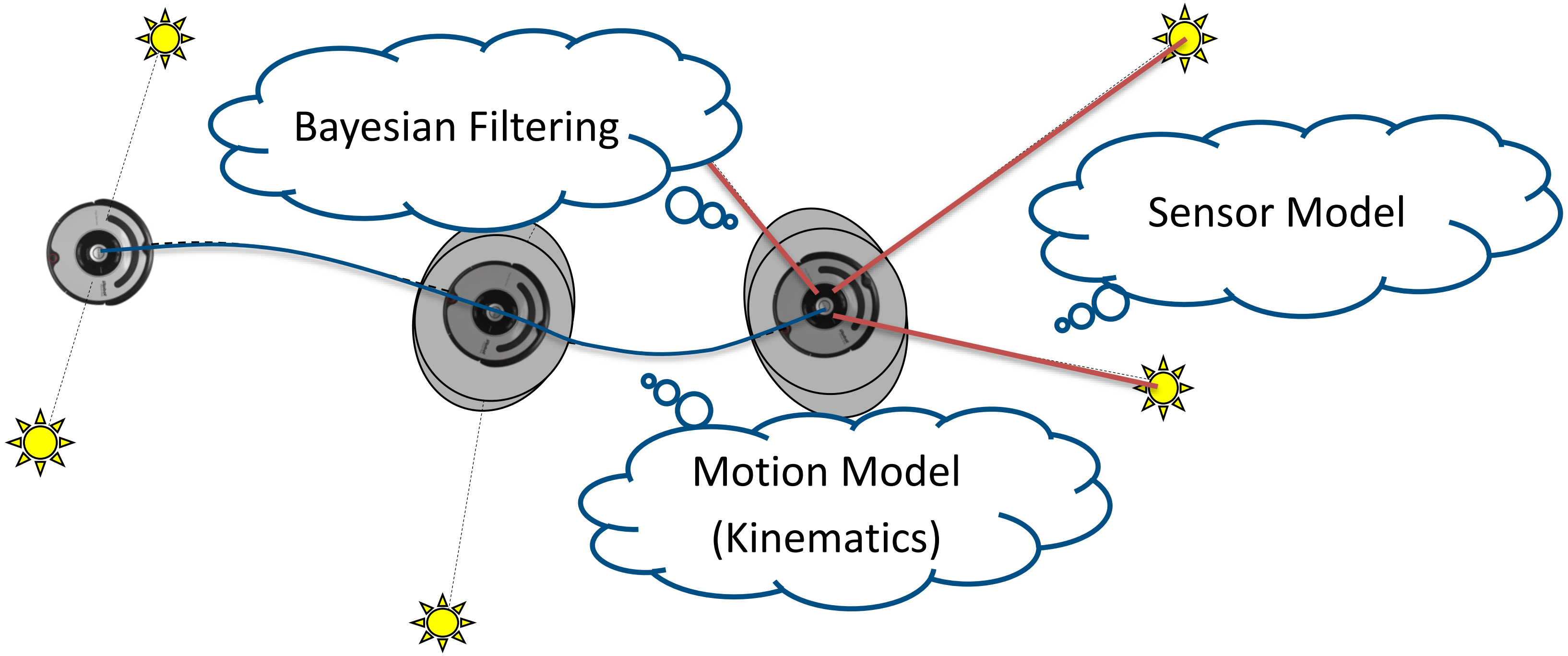


$P(z_1,z_2,z_3|x,m)$





Localization with Knowm Map

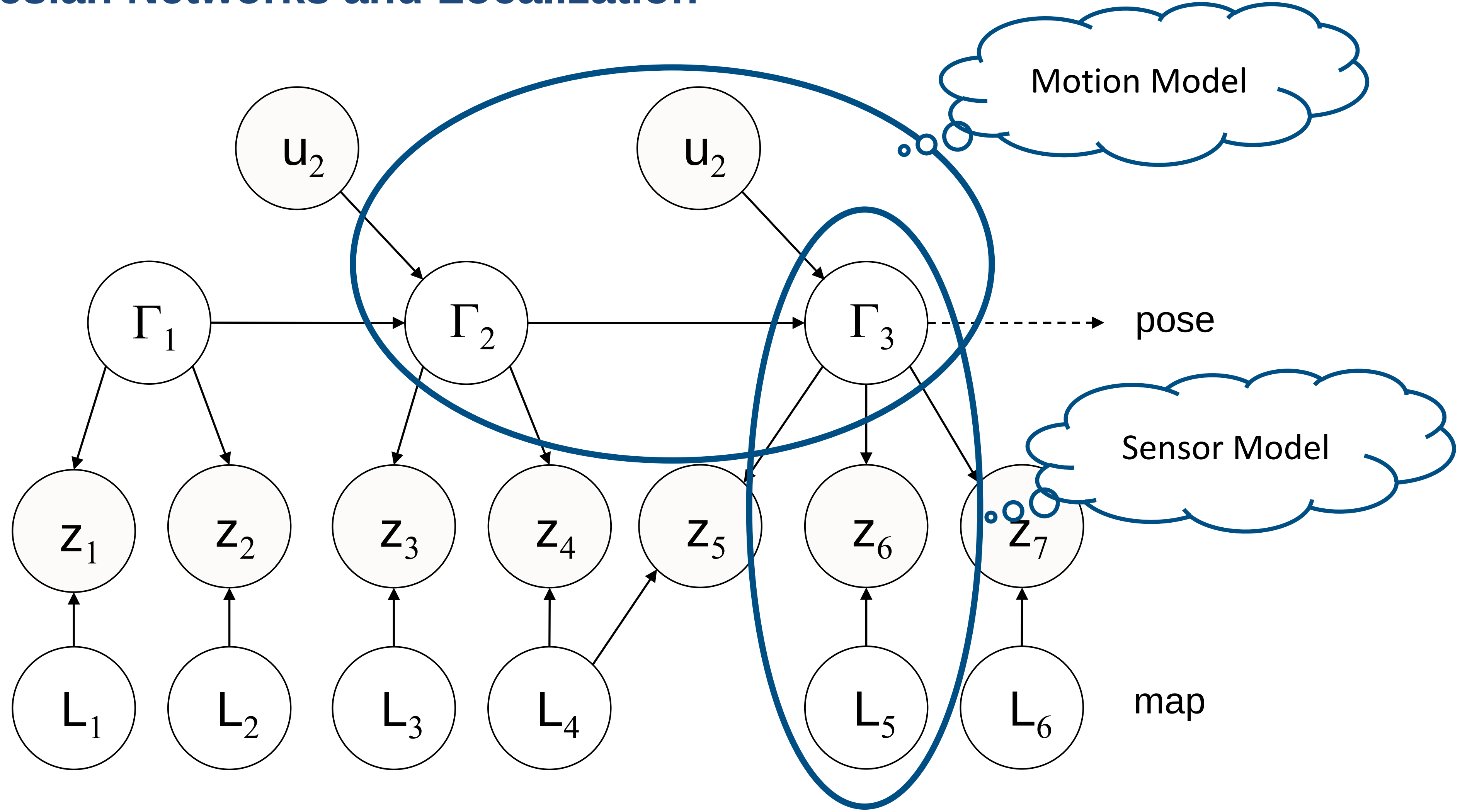


Bayesian Filtering

Sensor Model

Motion Model
(Kinematics)

Dynamic Bayesian Networks and Localization


Motion Model
u_2
u_2
Γ_1
Γ_2
Γ_3
pose
Sensor Model
z_1
z_2
z_3
z_4
z_5
z_6
z_7
L_1
L_2
L_3
L_4
L_5
L_6
map

$$\text{Filtering:} \quad p(\Gamma_t \mid Z_{1:t}, U_{1:t}, l_1, \ldots, l_N) = \iiint_{1:t-1} p(\Gamma_{1:t} \mid Z_{1:t}, U_{1:t}, l_1, \ldots, l_N)$$

Bayesian Filtering Framework

We want to compute an estimate of the posterios probabibility of robot state x_t

$$Bel(x_t) = P(x_t | u_1, z_1 \ \ldots, u_t, z_t, m)$$

from the stream of information about movement and sensors

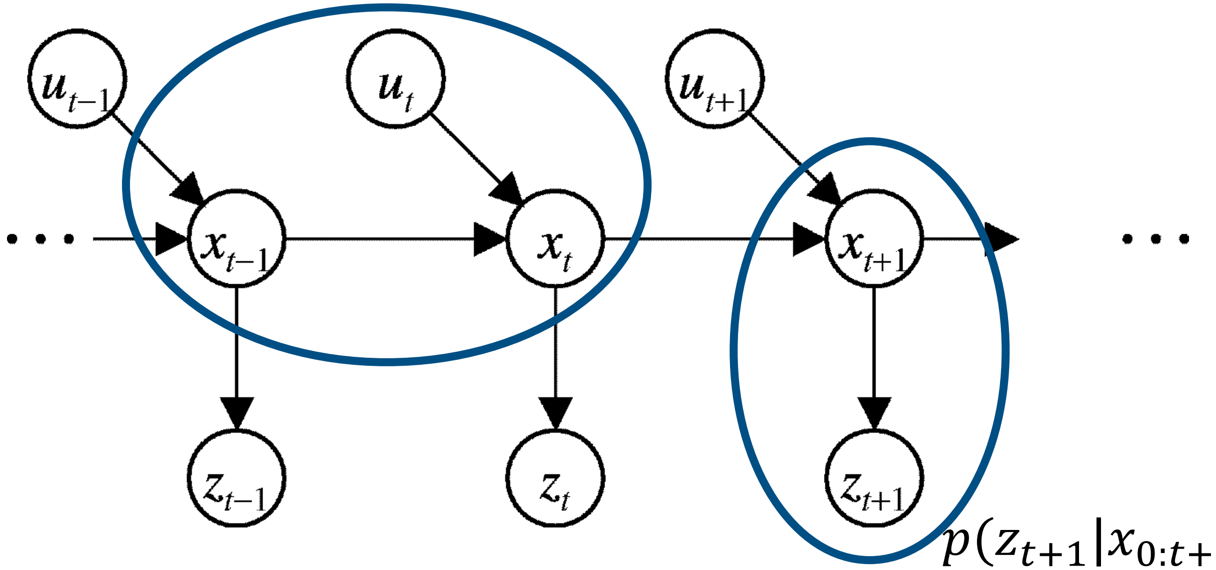
$$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$

In particular we assume known:

- The prior probability of the system state $P(x_0)$
- The motion model $P(x' | x, u)$
- The sensor model $P(z | x, m)$

Markov Assumptions

$$p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$



u_{t-1}
u_t
u_{t+1}
$\cdots$
x_{t-1}
x_t
x_{t+1}
$\cdots$
z_{t-1}
z_t
z_{t+1}

$$p(z_{t+1} | x_{0:t+1}, z_{1:t}, u_{1:t+1}) = p(z_{t+1} | x_{t+1})$$

Underlining assumption behind Bayes filtering:

- Perfect model, no approximation errors
- Static and stationary world
- Independent noise

Map is know as well, these are simplified here …

Bayes Filters

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t, m)$$

z = observation
u = action
x = state
m = map

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t, m) \, P(x_t \mid u_1, z_1, \ldots, u_t, m)$$

Markov $$= \eta \; P(z_t \mid x_t, m) \, P(x_t \mid u_1, z_1, \ldots, u_t, m)$$

Total prob. $$= \eta \; P(z_t \mid x_t, m) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}, m) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t, m) \, dx_{t-1}$$

Markov $$= \eta \; P(z_t \mid x_t, m) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots, u_t, m) \, dx_{t-1}$$

Markov $$= \eta P(z_t \mid x_t, m) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}, m) \, dx_{t-1}$$

$$\boxed{= \eta \; P(z_t \mid x_t, m) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}$$

Bayes Filter Algorithm

$$Bel(x_t|m) = \eta \; P(z_t|x_t, m) \; \int P(x_t|u_t, x_{t-1}, m) \; Bel(x_{t-1}|m) \; dx_{t-1}$$

How to represent such belief?

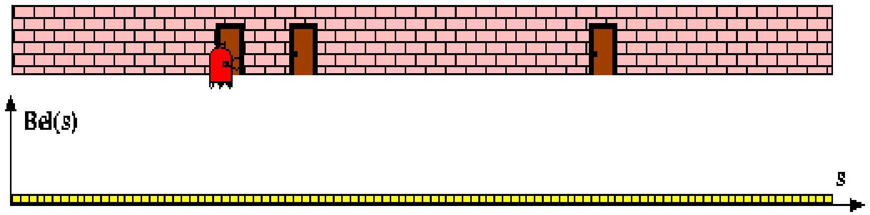
```
Algorithm Bayes_filter( Bel(x), d ):

    if d is a perceptual data item z then
        For all x do
            $Bel'(x) = P(z \mid x) Bel(x)$
        Normalize Bel'(x)
    else if d is an action data item u then
        For all x do
            $Bel'(x) = \int P(x \mid u, x') \; Bel(x') \; dx'$
    return Bel'(x)
```

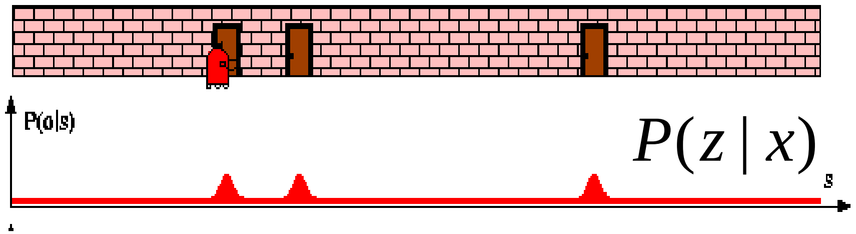
Based on such representation:

- Discrete filters
- Kalman filters
- Sigma-point filters
- Particle filters
- ...

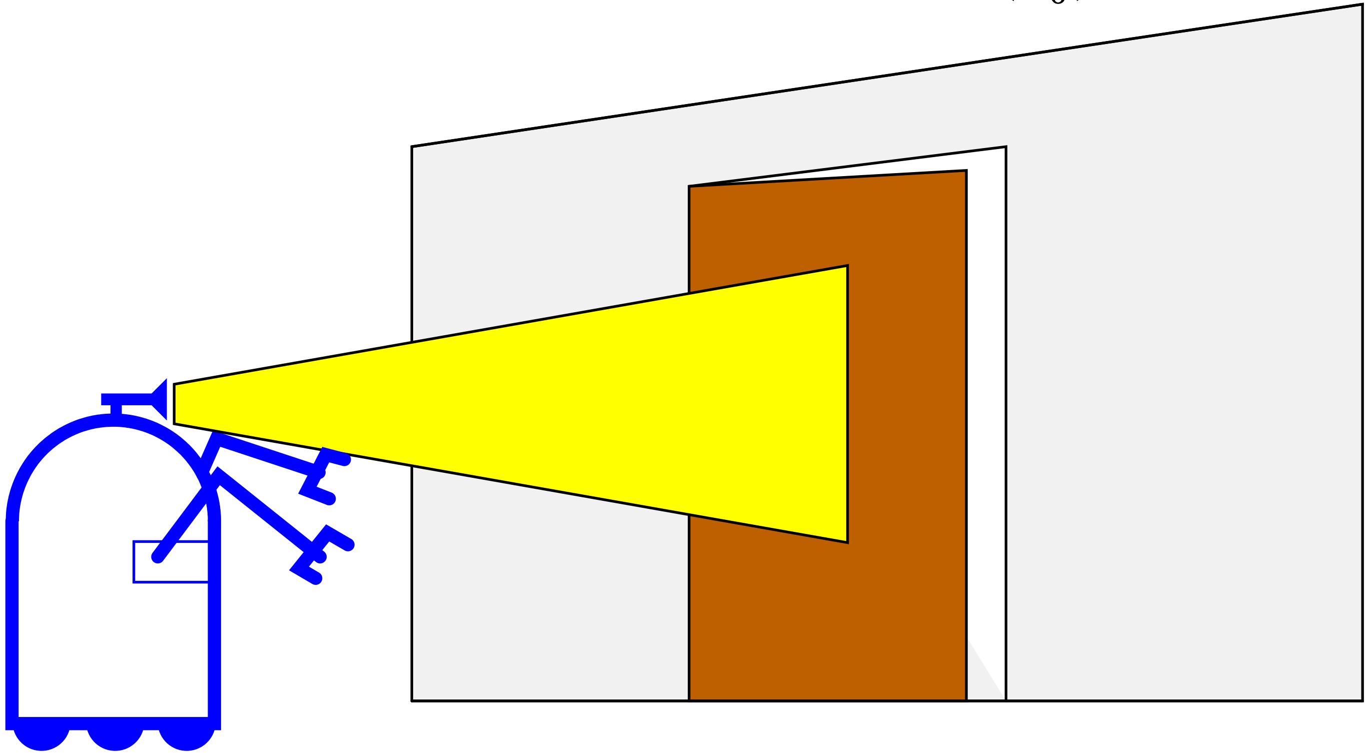
Piecewise Constant Approximation



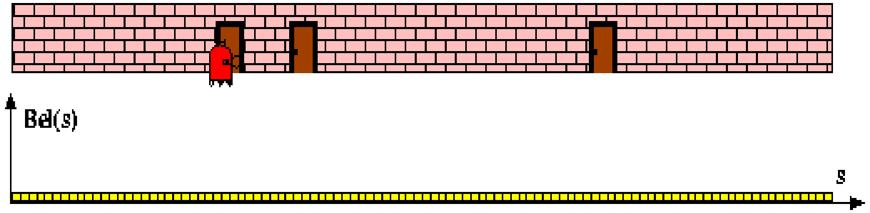
$Bel(x_0)$



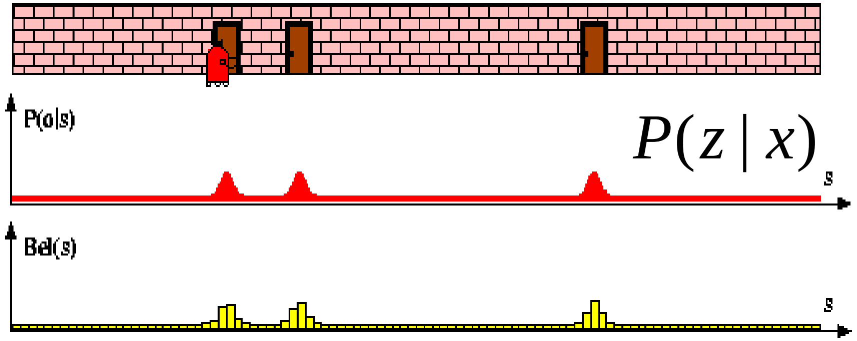
$P(z \mid x)$



Piecewise Constant Approximation

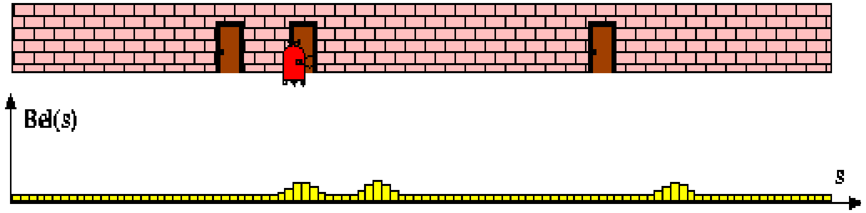


$$Bel(x_0)$$

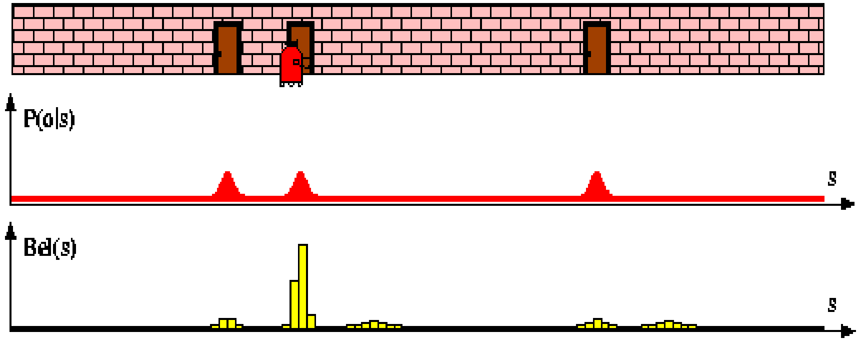


$$P(z \mid x)$$

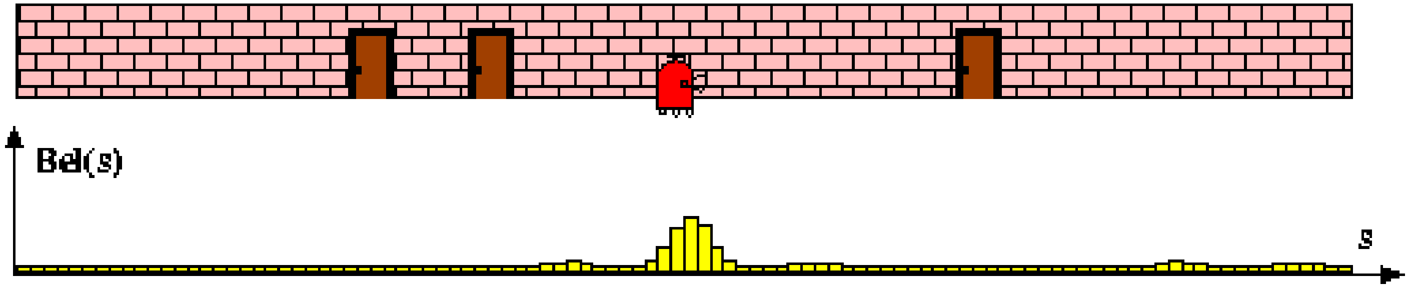
$$Bel'(x_0) = P(z \mid x) Bel(x_0)$$



$$Bel(x_1) = P(x_1 \mid x_0, u_1)$$

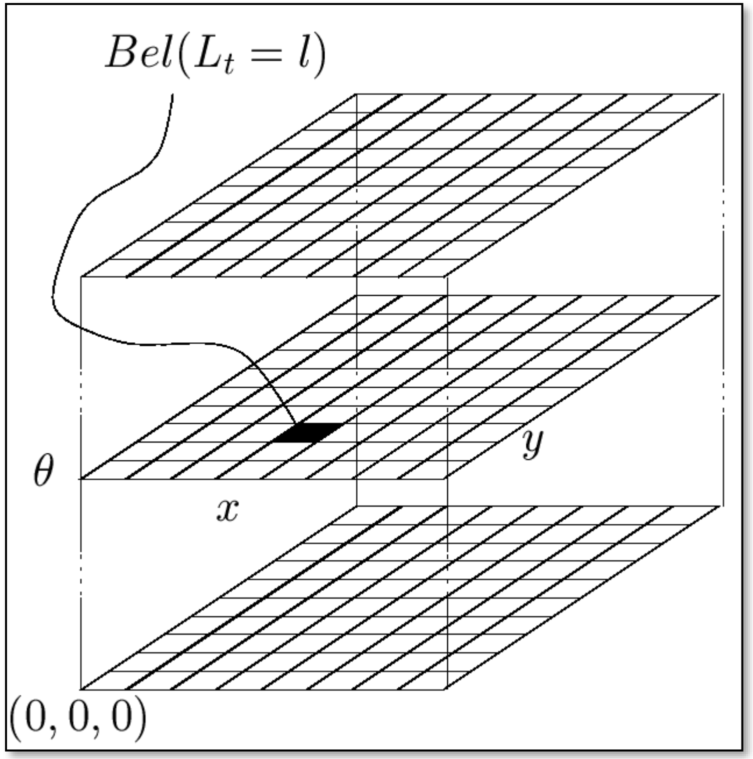


$$Bel'(x_1) = P(z \mid x_1) Bel(x_1)$$



Discrete Bayesian Filter Algorithm

```
Algorithm Discrete_Bayes_filter( Bel(x),d ):
    h=0
    If d is a perceptual data item z then
        For all x do
            $Bel'(x) = P(z \mid x) Bel(x)$
            $\eta = \eta + Bel'(x)$
        For all x do
            $Bel'(x) = \eta^{-1} Bel'(x)$
    Else if d is an action data item u then
        For all x do
            $Bel'(x) = \sum_{x'} P(x \mid u, x')\, Bel(x')$
    Return Bel'(x)
```



Tips and Tricks

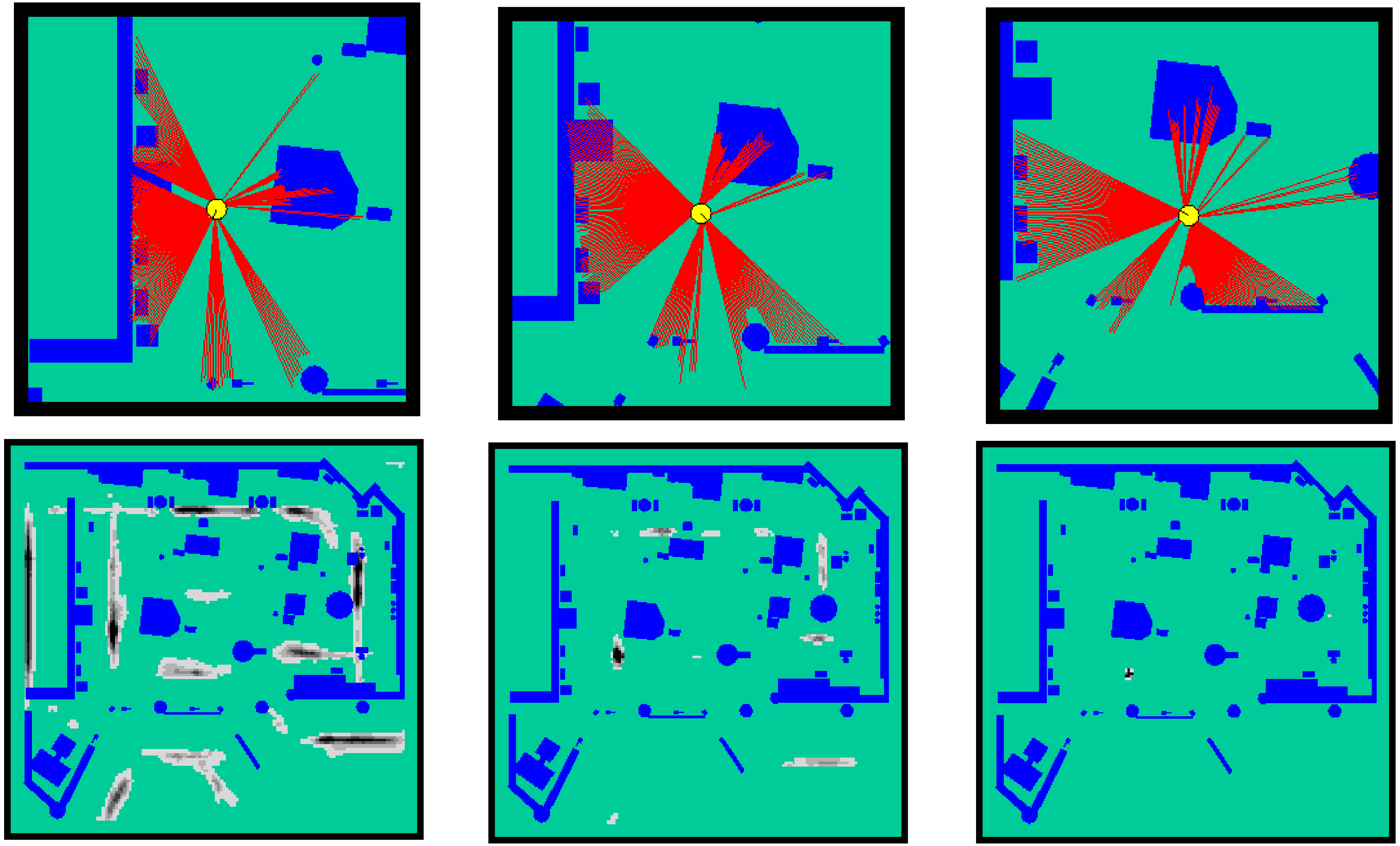
Belief update upon sensory input and normalization iterates over all cells

- When the belief is peaked (e.g., during position tracking), avoid updating irrelevant parts.
- Do not update entire sub-spaces of the state space and monitor whether the robot is de-localized or not by considering likelihood of observations given the active components

To update the belief upon robot motions, assumes a bounded Gaussian model to reduce the update from $O(n^2)$ to $O(n)$

- Update by shifting the data in the grid according to measured motion
- Then convolve the grid using a Gaussian Kernel.

Grid Based Localization



Bayes Filter Algorithm

$$Bel(x_t|m) = \eta \ P(z_t|x_t, m) \int P(x_t|u_t, x_{t-1}, m) \ Bel(x_{t-1}|m) \ dx_{t-1}$$

How to represent such belief?

Algorithm Bayes_filter(*Bel(x), d*):

```
if d is a perceptual data item z then
    For all x do
        $Bel'(x) = P(z \mid x) Bel(x)$
    Normalize Bel'(x)
else if d is an action data item u then
    For all x do
        $Bel'(x) = \int P(x \mid u, x') \ Bel(x') \ dx'$
return Bel'(x)
```

Based on such representation:

- Discrete filters ⬅
- Kalman filters
- Sigma-point filters
- Particle filters
- ...

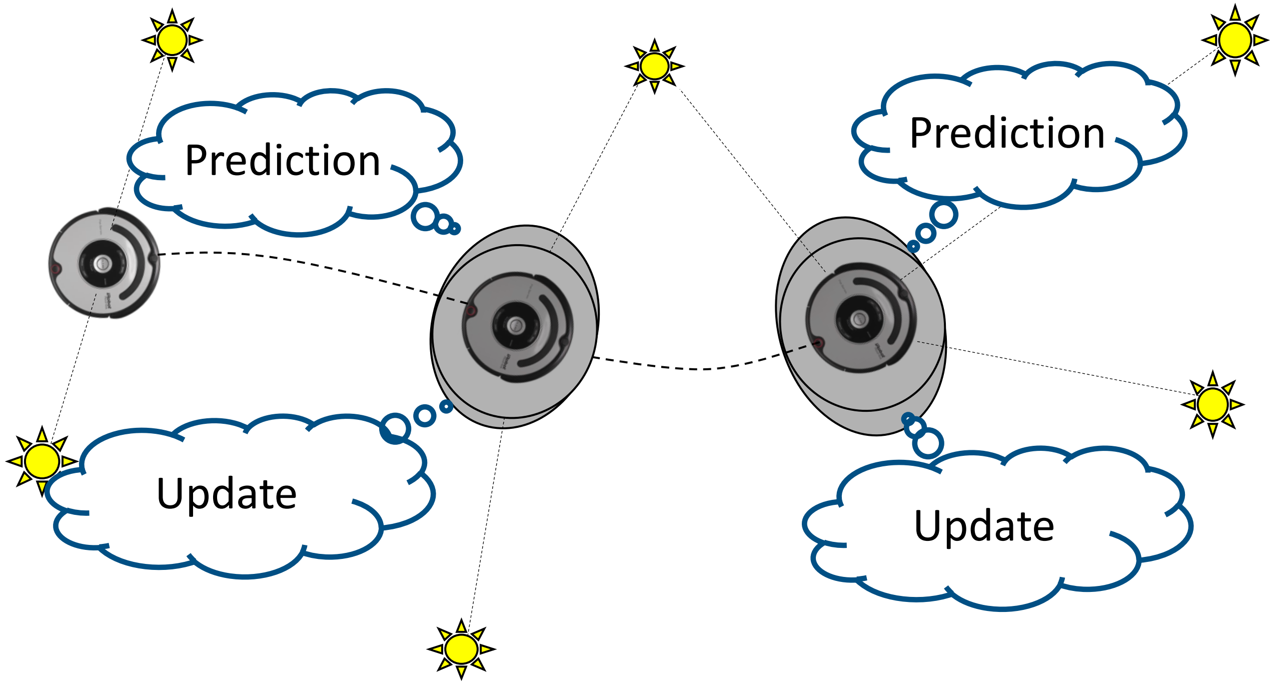
Bayes Filter Reminder

$$\boxed{Bel(x_t|m) = \eta\ P(z_t|x_t, m)\ \int P(x_t|u_t, x_{t-1}, m)\ Bel(x_{t-1}|m)\ dx_{t-1}}$$

Prediction: $\overline{Bel}(x_t|m) = \int p(x_t|u_t, x_{t-1}, m)\ Bel(x_{t-1}|m)\ dx_{t-1}$

Correction/Update: $Bel(x_t|m) = \ \eta p(z_t|x_t., m)\overline{Bel}(x_t|m)$

Localization with Knowm Map


Prediction
Update
Prediction
Update

Bayes Filter Reminder

$$\boxed{Bel(x_t|m) = \eta\ P(z_t|x_t,m)\ \int P(x_t|u_t,x_{t-1},m)\ Bel(x_{t-1}|m)\ dx_{t-1}}$$

Prediction: $\overline{Bel}(x_t|m) = \int p(x_t|u_t,x_{t-1},m)\ Bel(x_{t-1}|m)\ dx_{t-1}$

Correction/Update: $Bel(x_t|m) = \ \eta p(z_t|x_t,m)\overline{Bel}(x_t|m)$

Can we compute the integrals (η is an integral too) in closed form for continuos distributions?

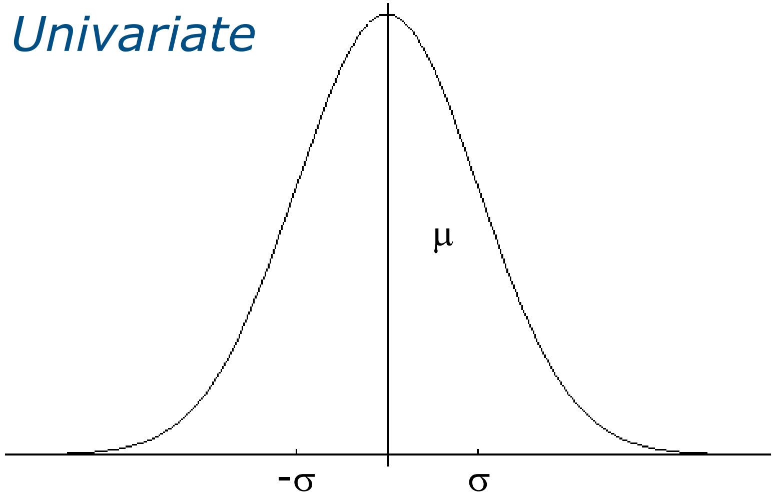
NO!

Is there any continuous distribution for which this is possible?

YES!

Gaussian Distribution

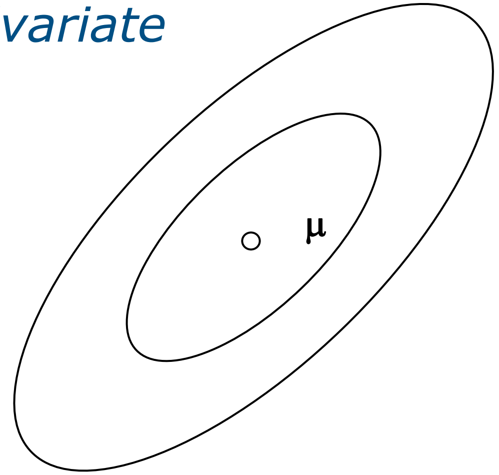
Univariate



$$p(x) \sim N(\mu,\sigma^2)$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Multivariate

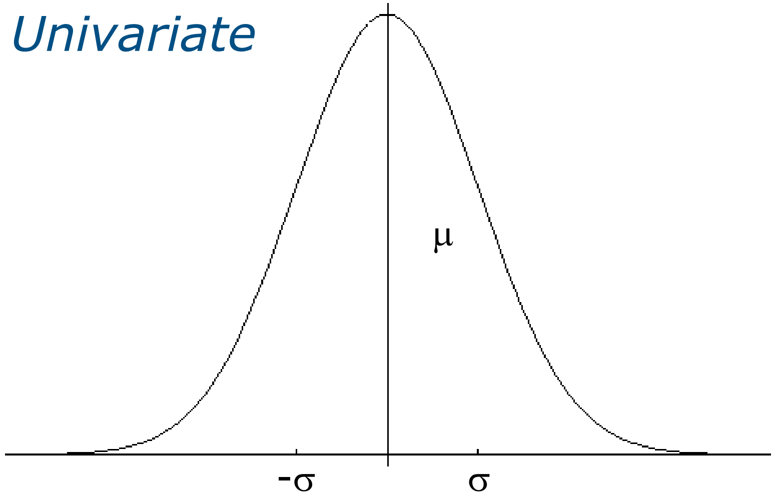


$$p(\mathbf{x}) \sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\left|\boldsymbol{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Properties of Gaussian Distribution

Univariate



$$p(x) \sim N(\mu, \sigma^2)$$

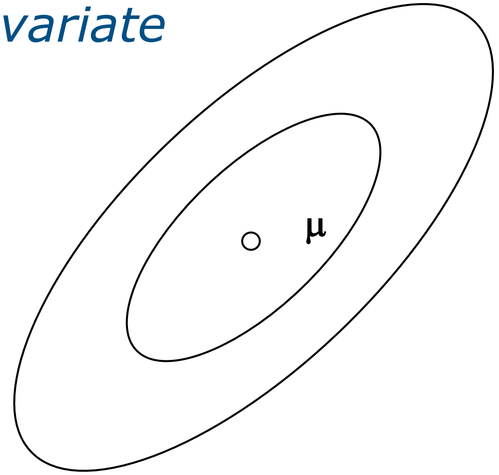
$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$\left.\begin{aligned} X &\sim N(\mu, \sigma^2) \\ Y &= aX + b \end{aligned}\right\} \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{aligned} X_1 &\sim N(\mu_1, \sigma_1^{\;2}) \\ X_2 &\sim N(\mu_2, \sigma_2^{\;2}) \end{aligned}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^{\;2}}{\sigma_1^{\;2} + \sigma_2^{\;2}} \mu_1 + \frac{\sigma_1^{\;2}}{\sigma_1^{\;2} + \sigma_2^{\;2}} \mu_2, \quad \frac{1}{\sigma_1^{\;-2} + \sigma_2^{\;-2}} \right)$$

Properties of Gaussian Distribution

Multivariate



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

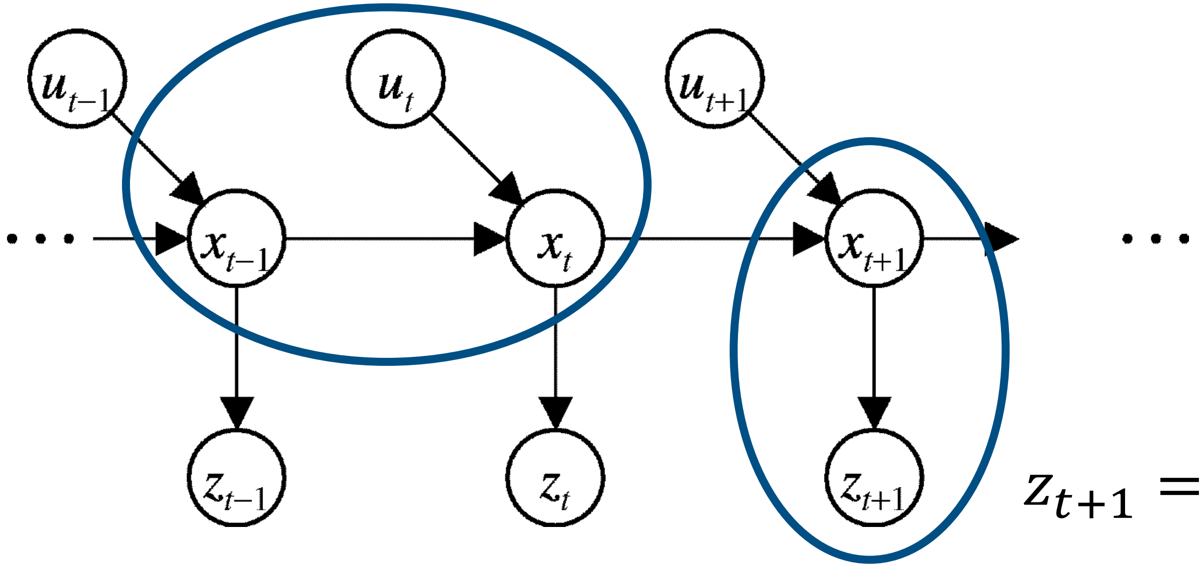
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

$$\left.\begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$

Discrete Time Kalman Filter

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

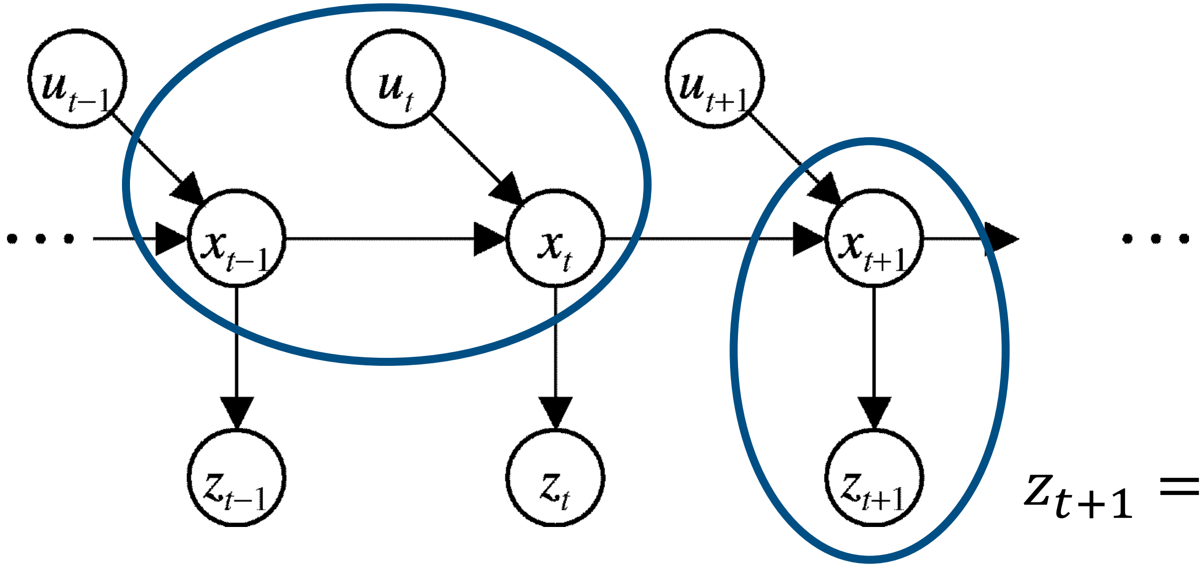


$$z_{t+1} = C_{t+1} x_{t+1} + \delta_{t+1}$$

- A_t (n x n) describes how state evolves from t-1 to t w/o controls or noise
- B_t (n x l) describes how control u_t changes the state from t-1 to t
- C_t (k x n) describes how to map the state x_t to an observation z_t
- ε_t δ_t random variables representing process and measurement noise assumed independent and normally distributed with covariance R_t and Q_t respectively.

Linear Gaussian Systems

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$



$$z_{t+1} = C_{t+1} x_{t+1} + \delta_{t+1}$$

Initial belief is normally distributed: $Bel(x_0) = N(\mu_0, \Sigma_0)$

Dynamics are linear function of state and control plus additive noise:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t \Rightarrow p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)$$

Observations are linear function of state plus additive noise:

$$z_t = C_t x_t + \delta_t \Rightarrow p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)$$

Linear Gaussian System: Prediction

Prediction:

$$\overline{Bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\sim N\left(x_t;\, A_t x_{t-1} + B_t u_t,\, R_t\right)} \cdot \underbrace{Bel(x_{t-1})}_{\sim N\left(x_{t-1};\, \mu_{t-1}, \Sigma_{t-1}\right)}\, dx_{t-1}$$

$$\overline{Bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}$$
$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{Bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Closed form
prediction step

Linear Gaussian System: Observation

Update: $$Bel(x_t) = \underbrace{\eta}_{} \quad \underbrace{p(z_t \mid x_t)}_{\sim N(z_t; C_t x_t, Q_t)} \quad \cdot \quad \underbrace{\overline{bel}(x_t)}_{\sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t)}$$

$$Bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$Bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Closed form update step

Kalman Filter Algorithm

Algorithm Kalman_filter(μ_{t-1}, Σ_{t-1}, u_t, z_t):

Prediction:

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

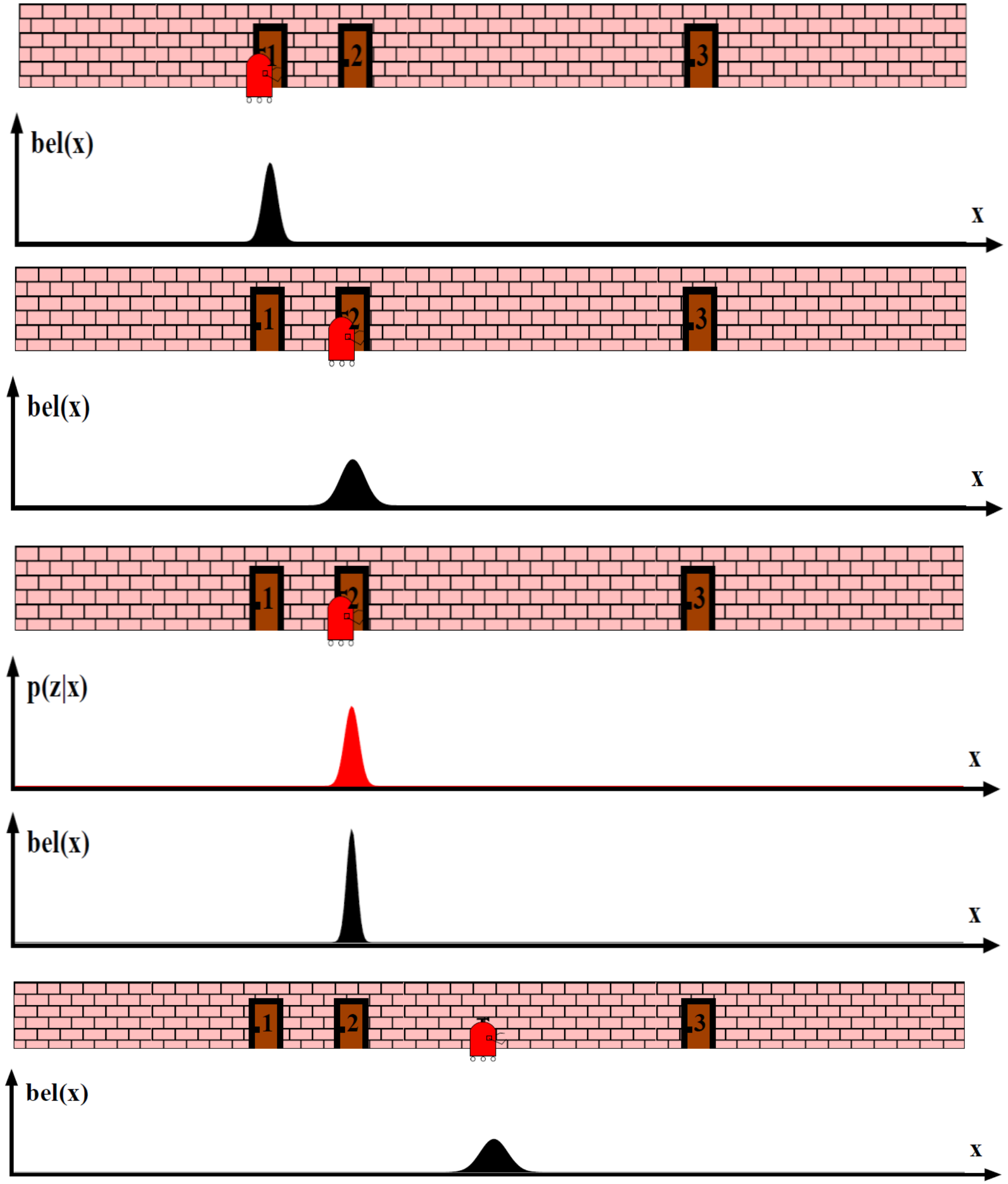
Correction:

$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

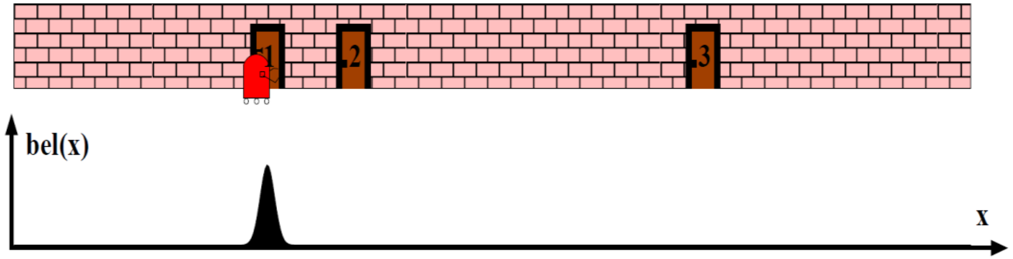
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

Return μ_t, Σ_t


1
2
3
bel(x)
x
bel(x)
x
p(z|x)
x
bel(x)
x
bel(x)
x

Kalman Filter Algorithm



Algorithm Kalman_filter(μ_{t-1}, Σ_{t-1}, u_t, z_t):

Prediction:

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

Correction:

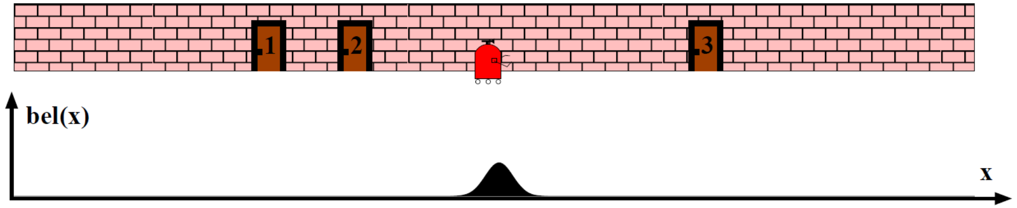
$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

Return μ_t, Σ_t

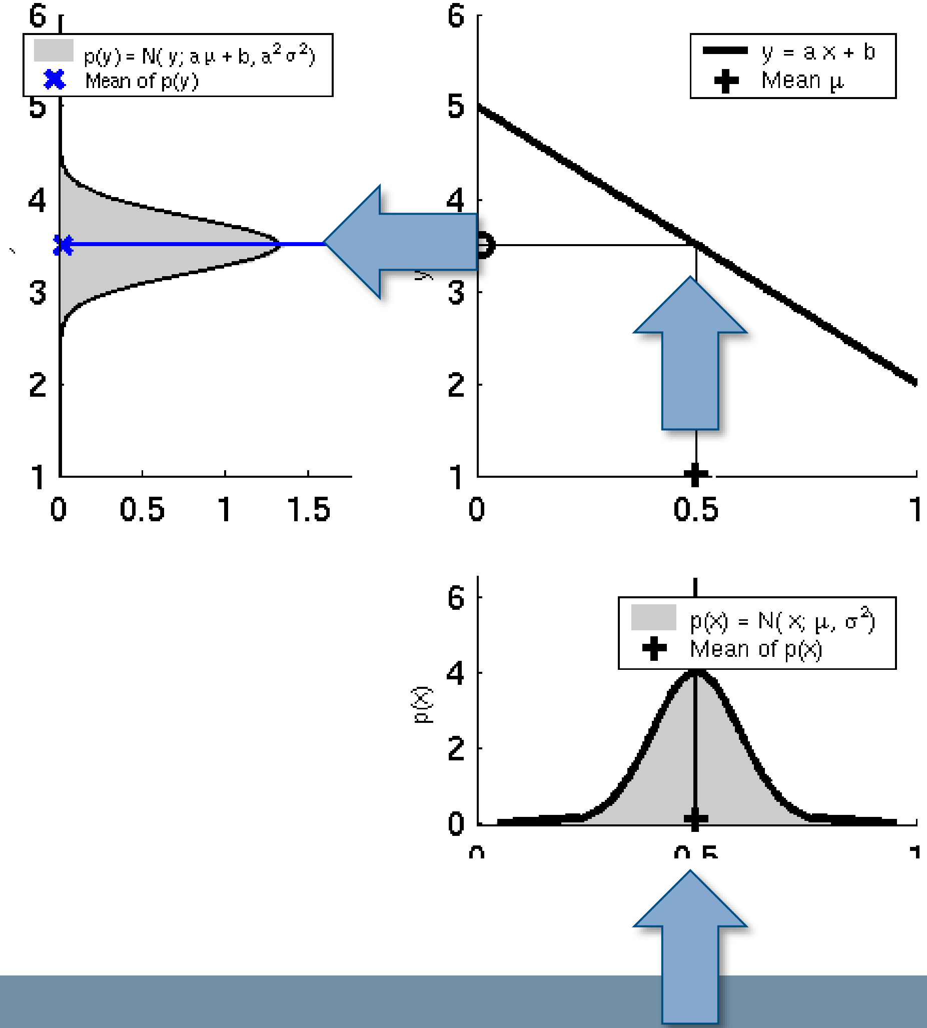
- Polynomial in measurement dimensionality k and state dimensionality n: $O(k^{2.376} + n^2)$
- Optimal for linear Gaussian systems ☺
- Most robotics systems are nonlinear ☹
- It represents unimodal distributions ☹



How to Deal with Non Linear Dynamic Systems?

Gaussian noise in linear systems

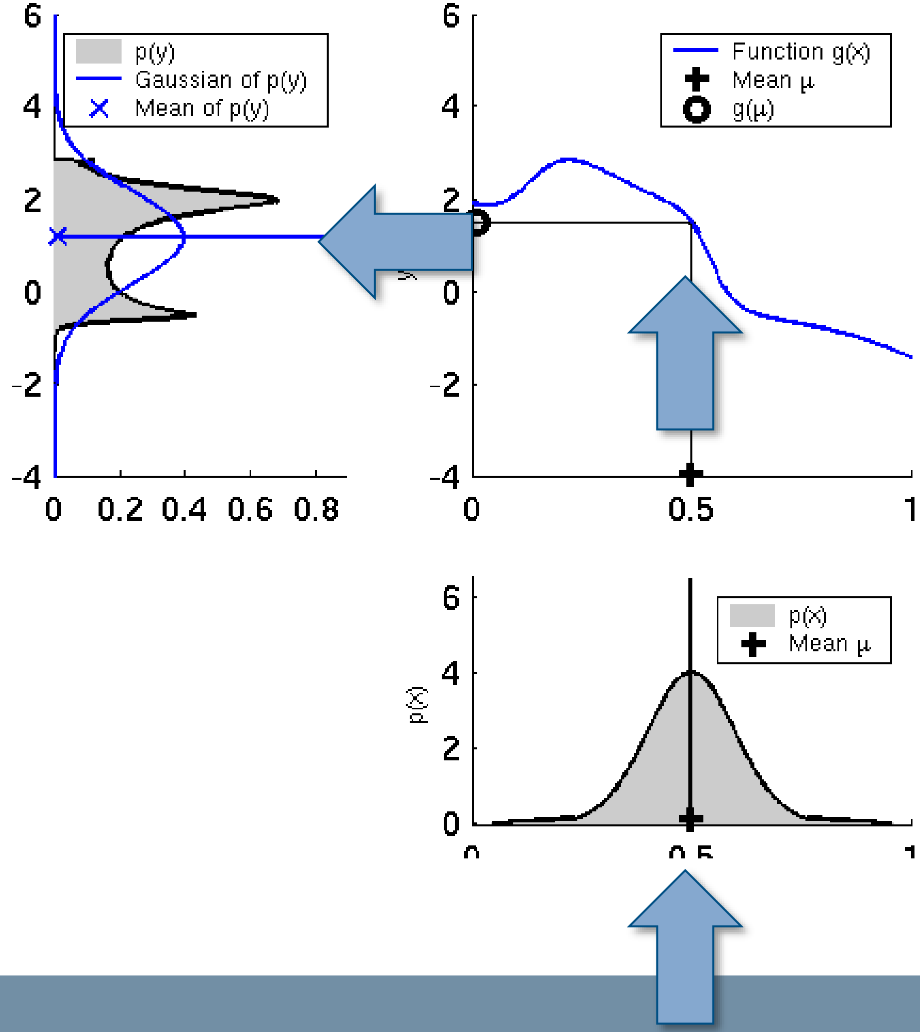
$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$
$$z_t = C_t x_t + \delta_t$$



How to Deal with Non Linear Dynamic Systems?

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$
$$z_t = h(x_t)$$



Extended Kalman Filter (First order Taylor approximation)

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

Prediction:

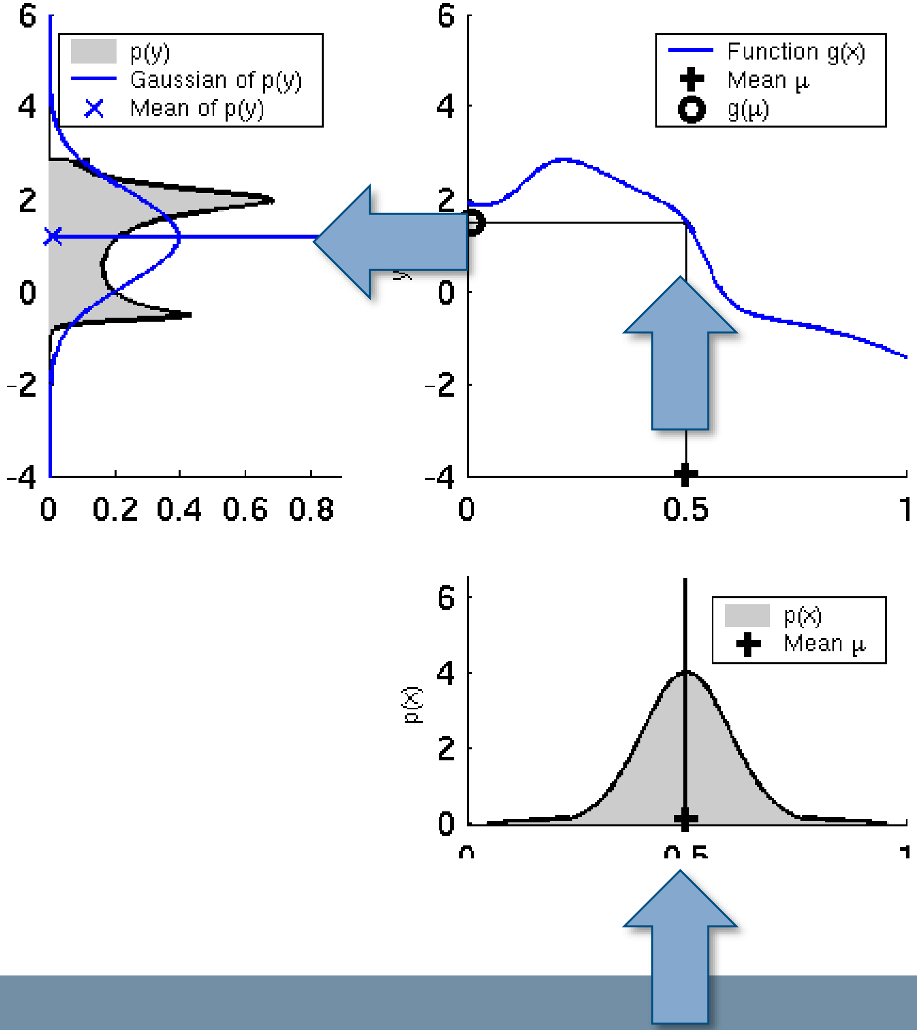
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

Correction

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



Extended Kalman Filter (First order Taylor approximation)

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$
$$z_t = h(x_t)$$

Prediction:

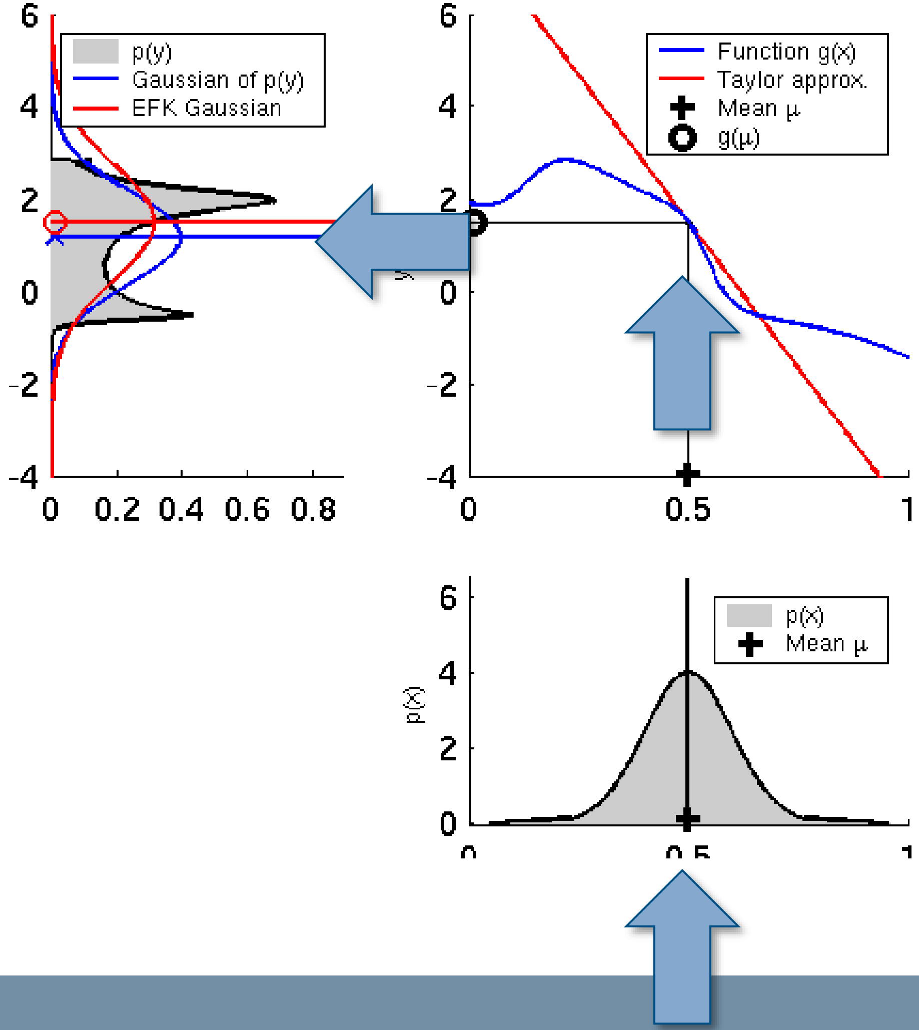
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

Correction

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



Extended Kalman Filter (First order Taylor approximation)

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$
$$z_t = h(x_t)$$

Prediction:

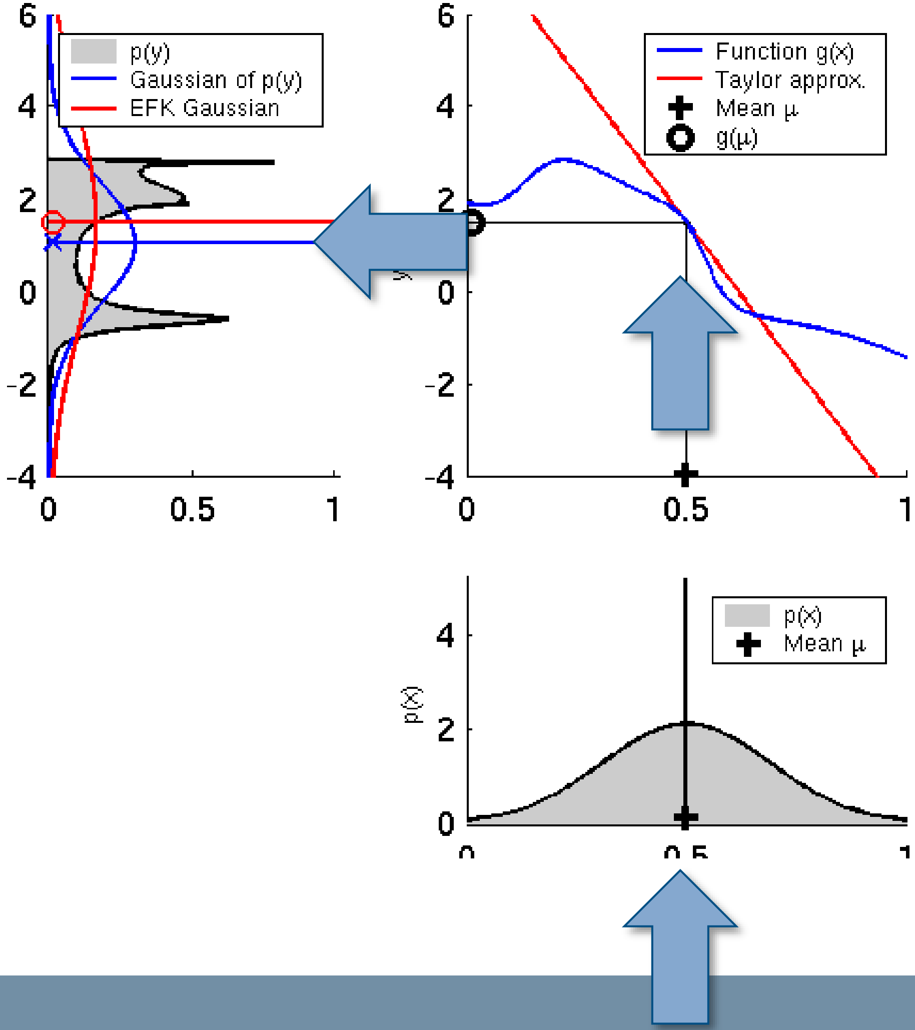
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

Correction

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



Extended Kalman Filter (First order Taylor approximation)

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$
$$z_t = h(x_t)$$

Prediction:

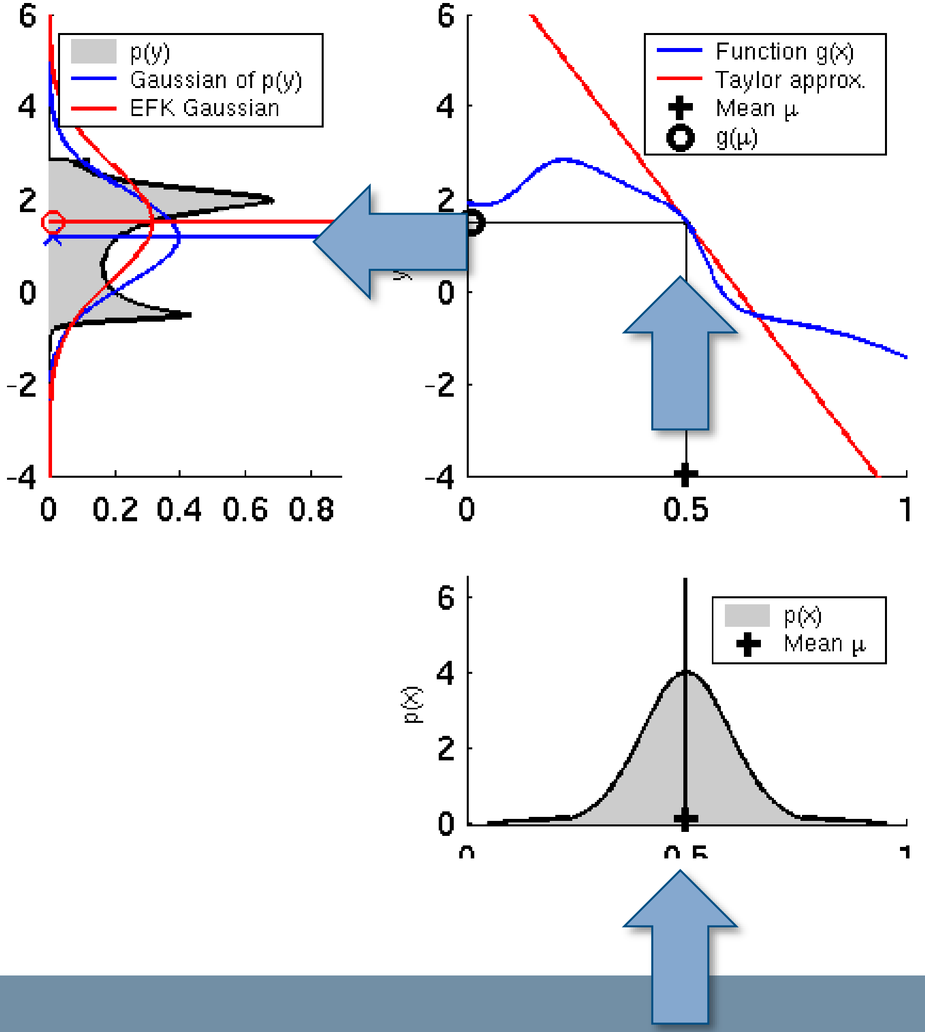
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

Correction

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



Extended Kalman Filter (First order Taylor approximation)

Gaussian noise in non-linear systems

$$x_t = g(u_t, x_{t-1})$$
$$z_t = h(x_t)$$

Prediction:

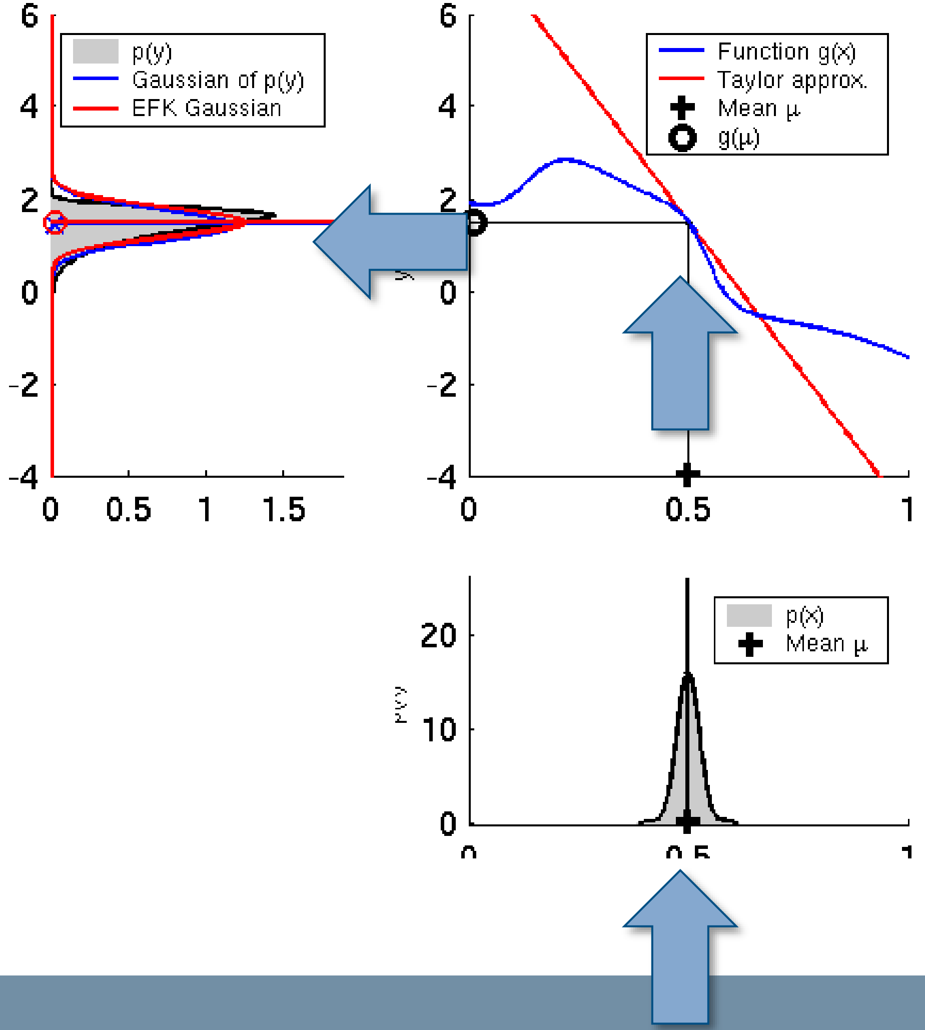
$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

Correction

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



EKF Algorithm

Extended_Kalman_filter(μ_{t-1}, Σ_{t-1}, u_t, z_t):

$$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} \qquad H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t}$$

Prediction:

$$\bar{\mu}_t = g(u_t, \mu_{t-1}) \quad \longleftrightarrow \quad \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t \quad \longleftrightarrow \quad \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

Linear form equations

Correction:

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1} \quad \longleftrightarrow \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t)) \quad \longleftrightarrow \quad \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t \quad \longleftrightarrow \quad \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

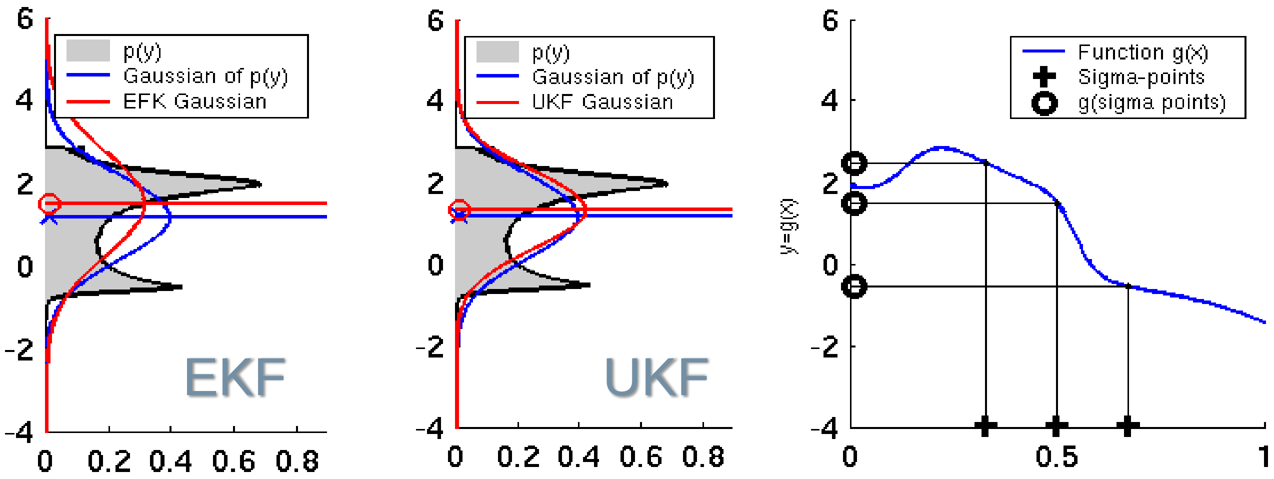
Return μ_t, Σ_t

Extended Kalman Filter Equations

EKF and Friends

Extended Kalman Filter:

- Polynomial in measurement k and state n dimensionality: $O(k^{2.376} + n^2)$
- Not optimal and can diverge if nonlinearities are large!
- Works surprisingly well even when all assumptions are violated!
- There are possible alternative like the Unscented Kalman Transform ...



Bayes Filter Algorithm

$$Bel(x_t|m) = \eta \; P(z_t|x_t, m) \; \int P(x_t|u_t, x_{t-1}, m) \; Bel(x_{t-1}|m) \; dx_{t-1}$$

How to represent such belief?

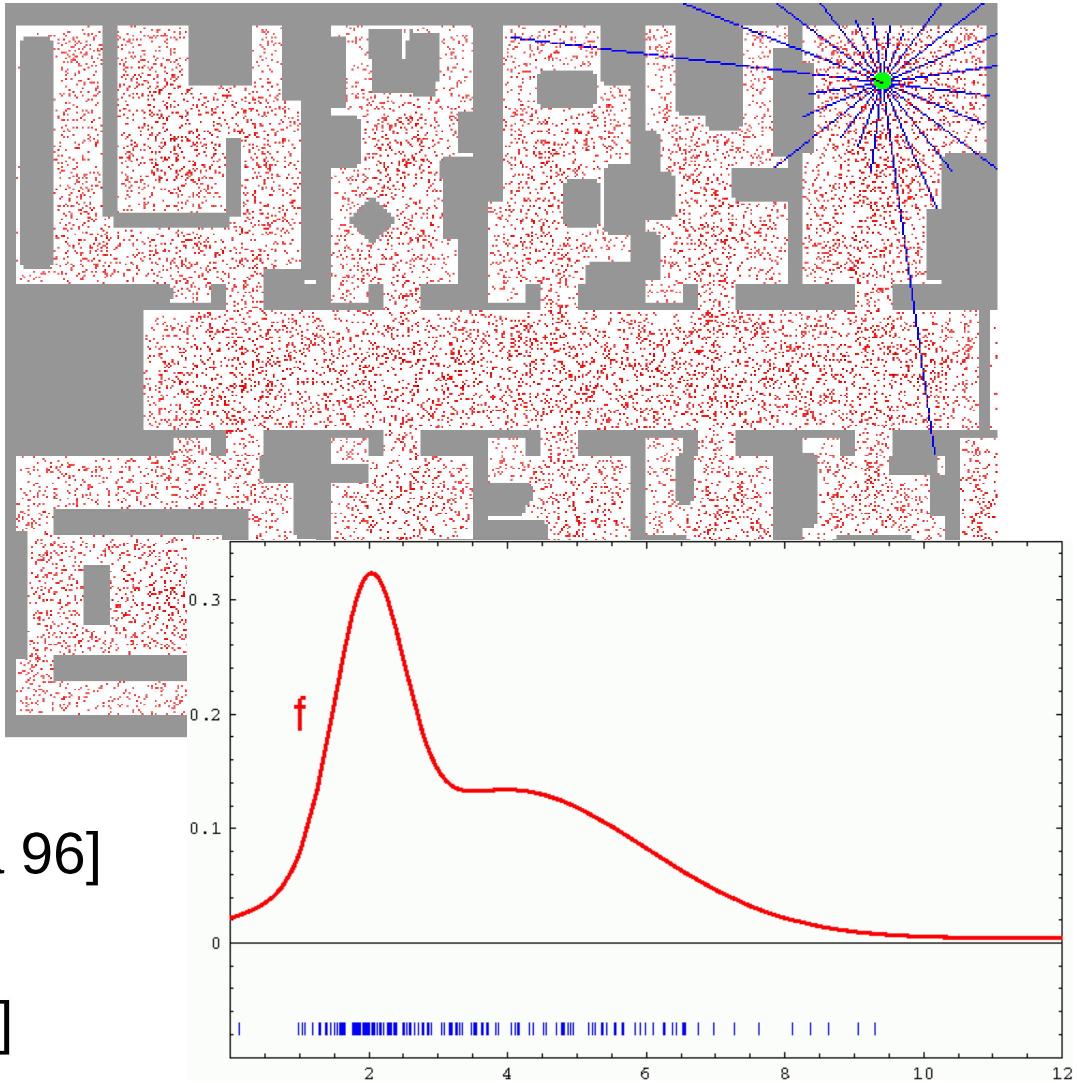
```
Algorithm Bayes_filter( Bel(x), d ):

        if d is a perceptual data item z then
            For all x do
                $Bel'(x) = P(z \mid x) Bel(x)$
            Normalize Bel'(x)
        else if d is an action data item u then
            For all x do
                $Bel'(x) = \int P(x \mid u, x') \; Bel(x') \; dx'$
        return Bel'(x)
```

Based on such representation:

- Discrete filters
- Kalman filters
- Sigma-point filters
- Particle filters
- ...

Particle Filters



Represent belief by random samples

Estimation of non-Gaussian, nonlinear processes

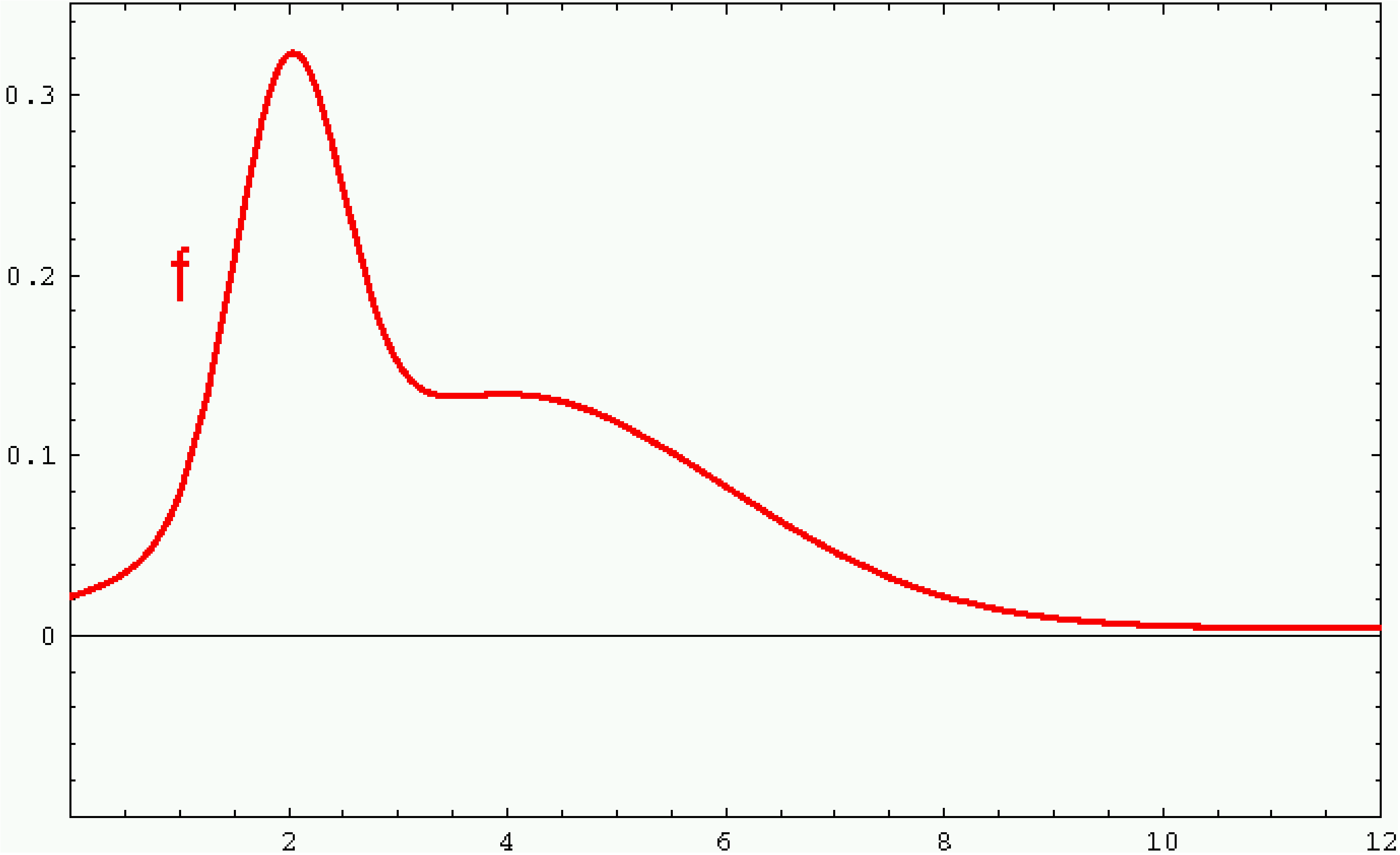
- Monte Carlo filter
- Survival of the fittest
- Condensation
- Bootstrap filter
- Particle filter

Filtering: [Rubin, 88], [Gordon et al., 93], [Kitagawa 96]

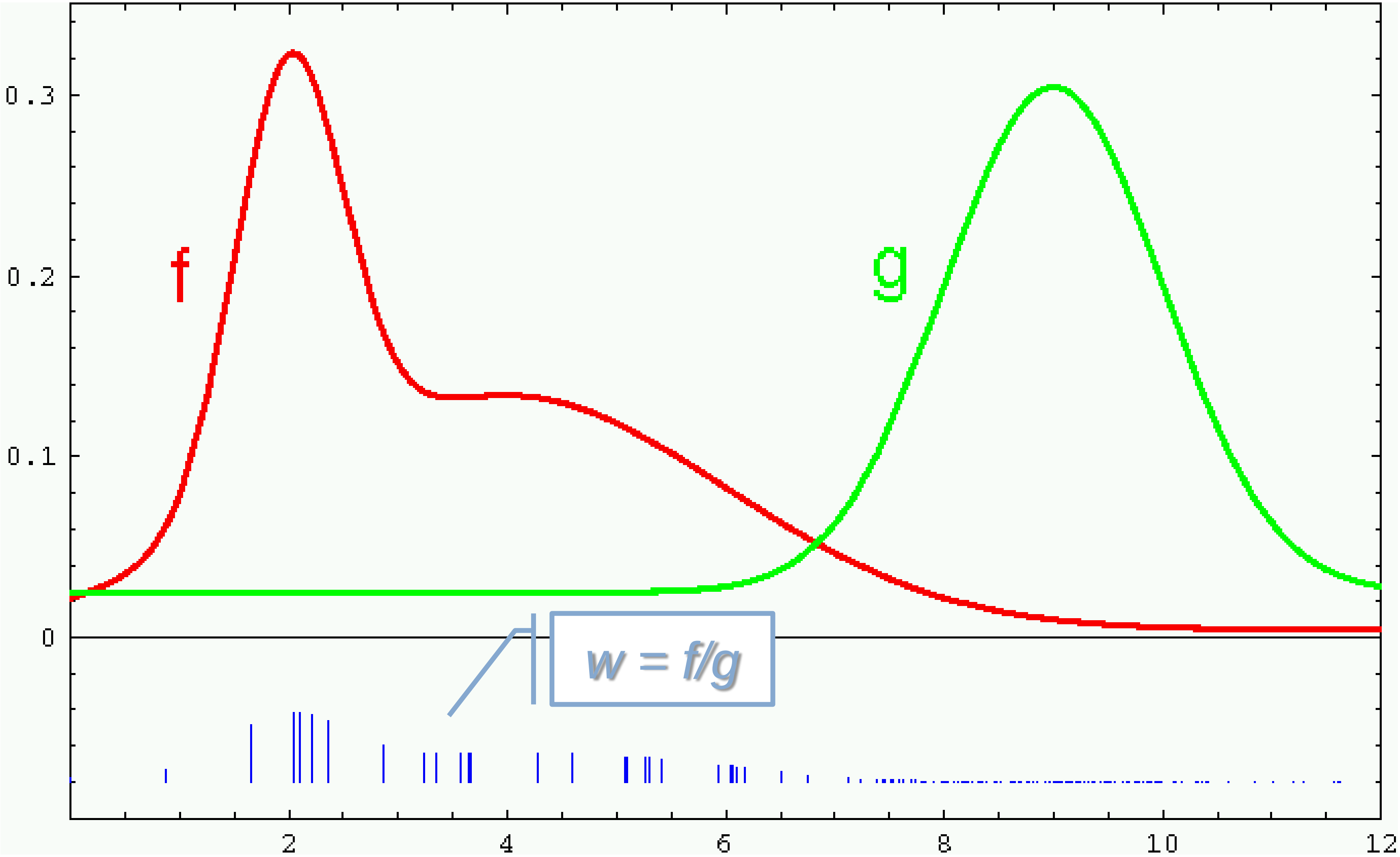
Computer vision: [Isard and Blake 96, 98]

Dynamic Bayesian Networks: [Kanazawa et al., 95]

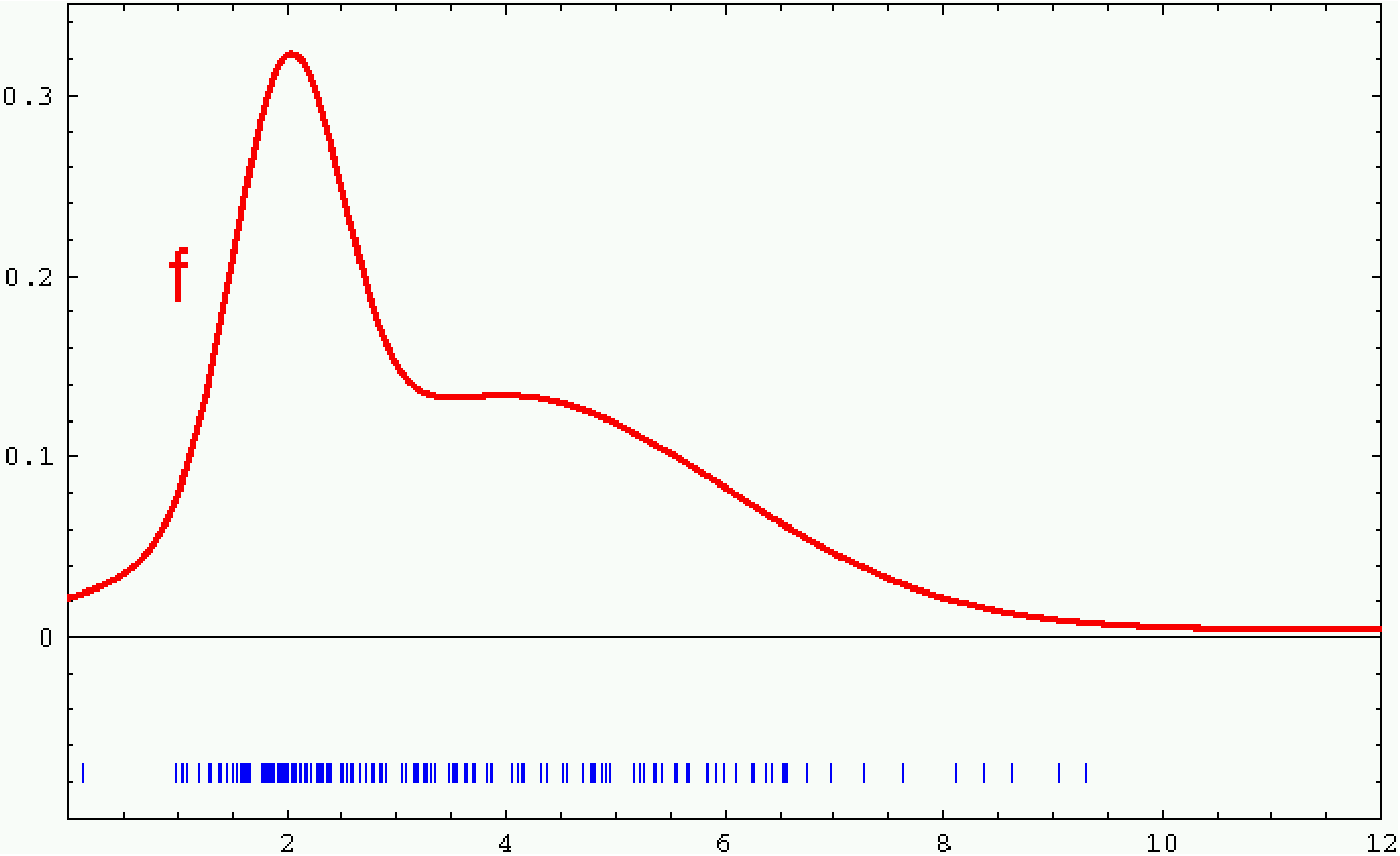
Importance Resampling



Importance Resampling



Importance Resampling (with smoothing)



Particle Filter Algorithm

$$Bel(x_t) = \eta \underbrace{p(z_t \mid x_t) \overbrace{\int p(x_t \mid x_{t-1}, u_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}^{g}}_{f}$$

draw x^i_{t-1} from $Bel(x_{t-1})$

draw x^i_t from $p(x_t \mid x^i_{t-1}, u_{t-1})$

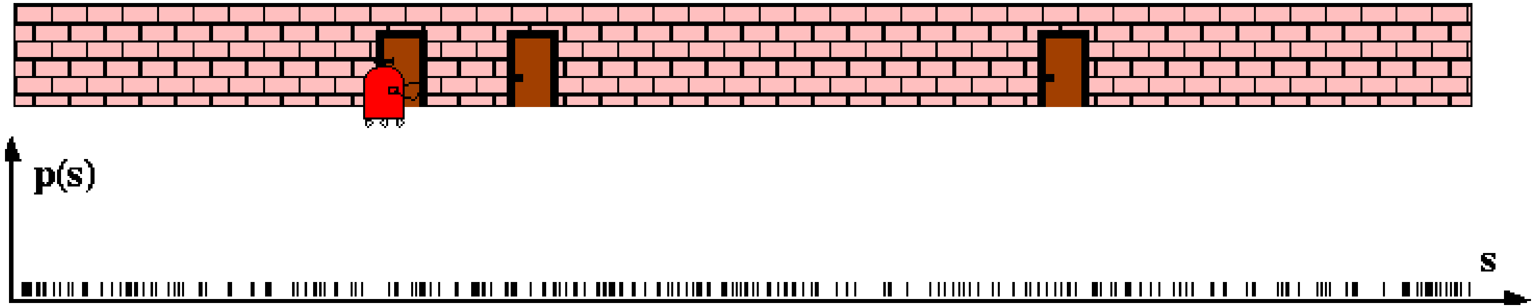
Importance factor for x^i_t:

$$\begin{aligned} w^i_t &= \frac{\text{target distribution}}{\text{proposal distribution}} \\ &= \frac{\eta \, p(z_t \mid x_t) \, p(x_t \mid x_{t-1}, u_{t-1}) \, Bel(x_{t-1})}{p(x_t \mid x_{t-1}, u_{t-1}) \, Bel(x_{t-1})} \\ &\propto p(z_t \mid x_t) \end{aligned}$$

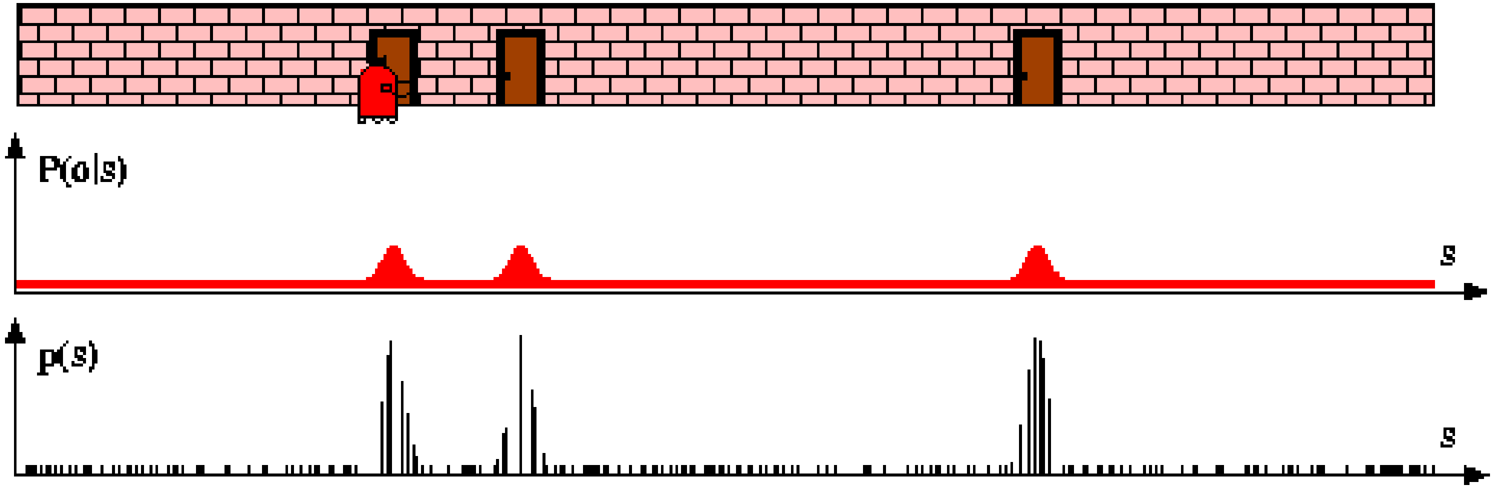
Particle Filter Algorithm

```
Algorithm particle_filter($S_{t-1}, u_{t-1}, z_t$):
        $S_t = \emptyset, \quad \eta = 0$
For $i = 1 \dots n$                                        Generate new samples
        Sample index $j(i)$ from the discrete distribution given by $w_{t-1}$
        Sample $x_t^i$ from $p(x_t \mid x_{t-1}, u_{t-1})$ using $x_{t-1}^{j(i)}$ and $u_{t-1}$
        $w_t^i = p(z_t \mid x_t^i)$                        Compute importance weight
        $\eta = \eta + w_t^i$                              Update normalization factor
        $S_t = S_t \cup \{< x_t^i, w_t^i >\}$              Insert
For $i = 1 \dots n$
        $w_t^i = w_t^i / \eta$                             Normalize weights
```

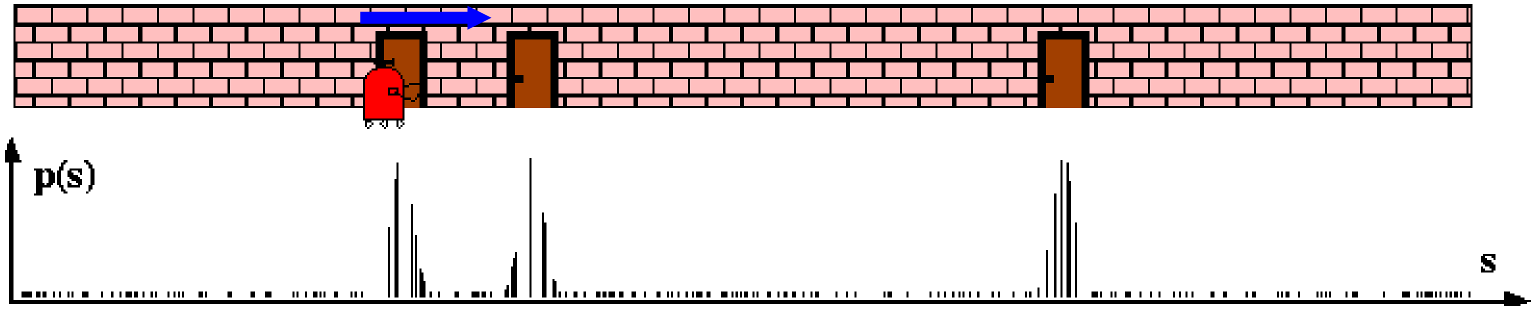
Sensor Information: Importance Sampling



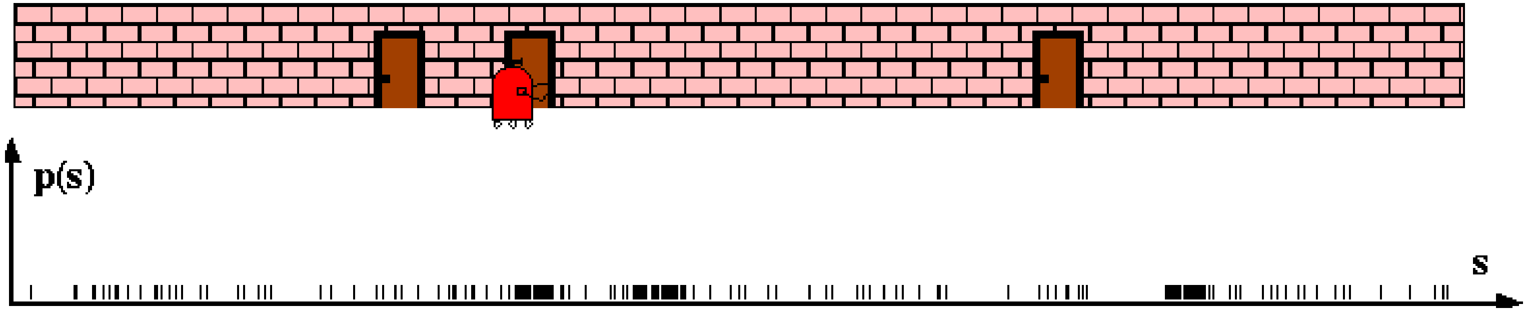
$$Bel(x) \leftarrow \alpha\; p(z\,|\,x)\, Bel^{-}(x) \qquad w \leftarrow \frac{\alpha\; p(z\,|\,x)\, Bel^{-}(x)}{Bel^{-}(x)} = \alpha\; p(z\,|\,x)$$



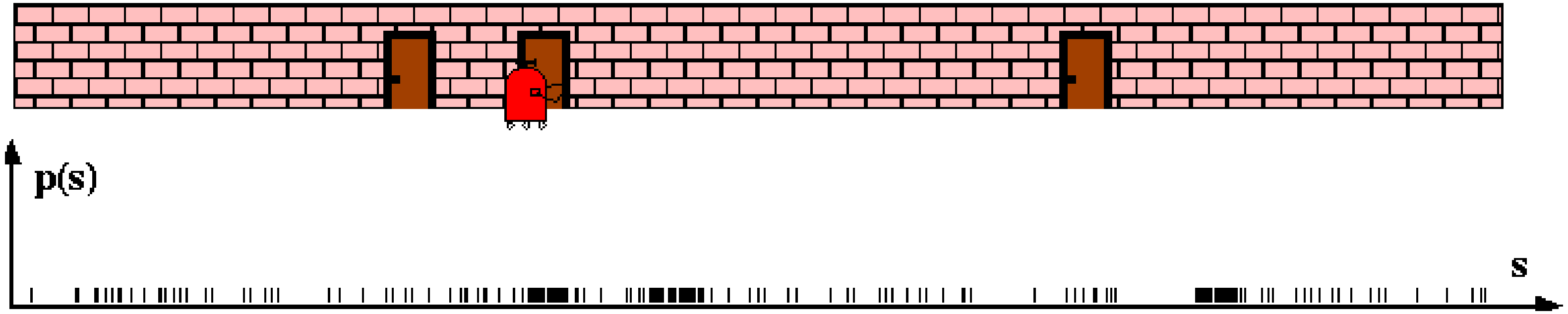
Sensor Information: Importance Sampling



$$Bel^{-}(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\ \mathrm{d}x'$$



Sensor Information: Importance Sampling



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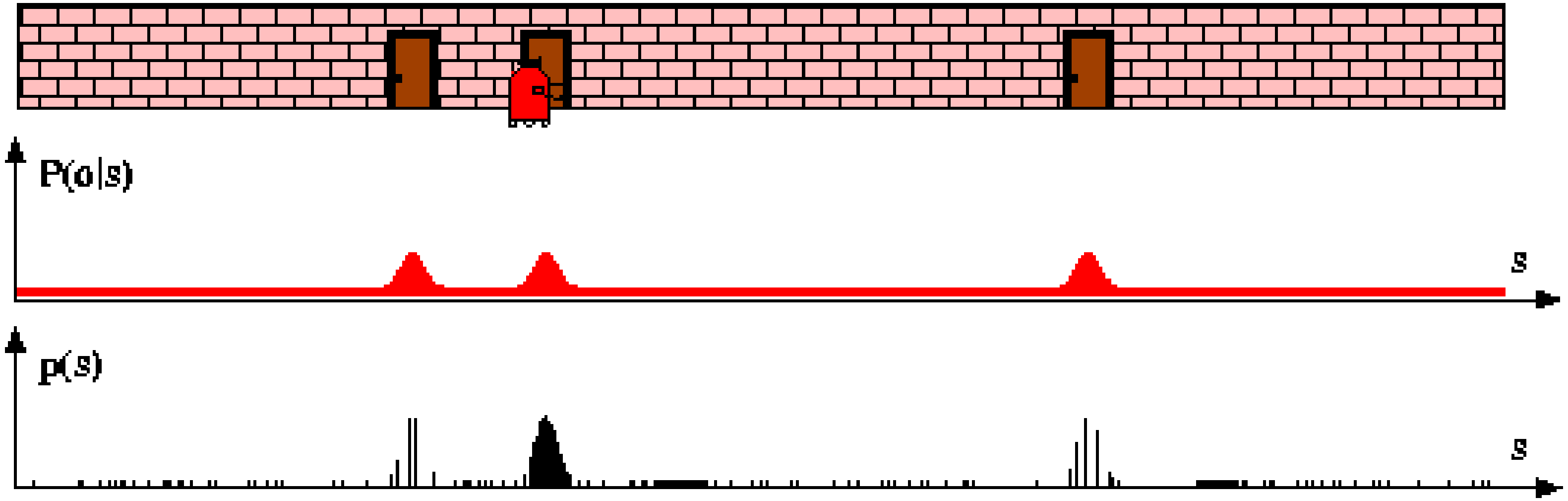
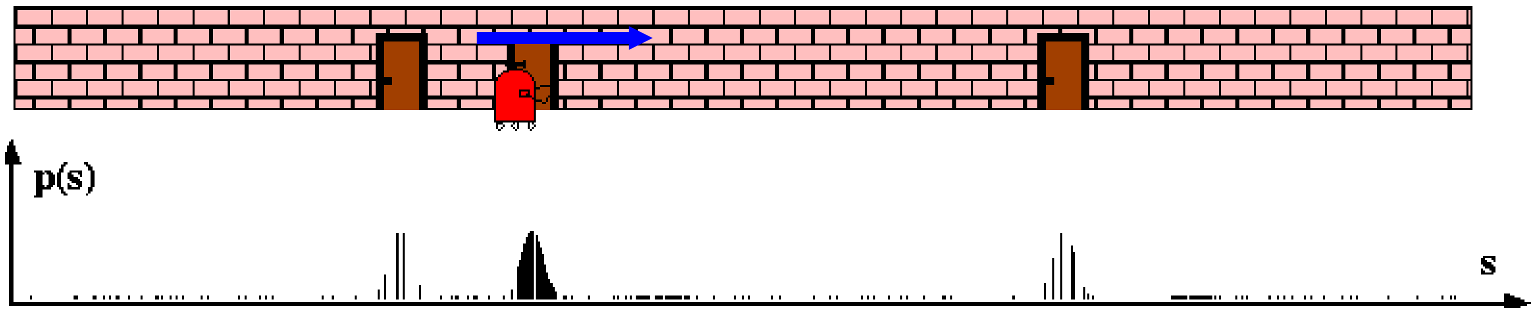
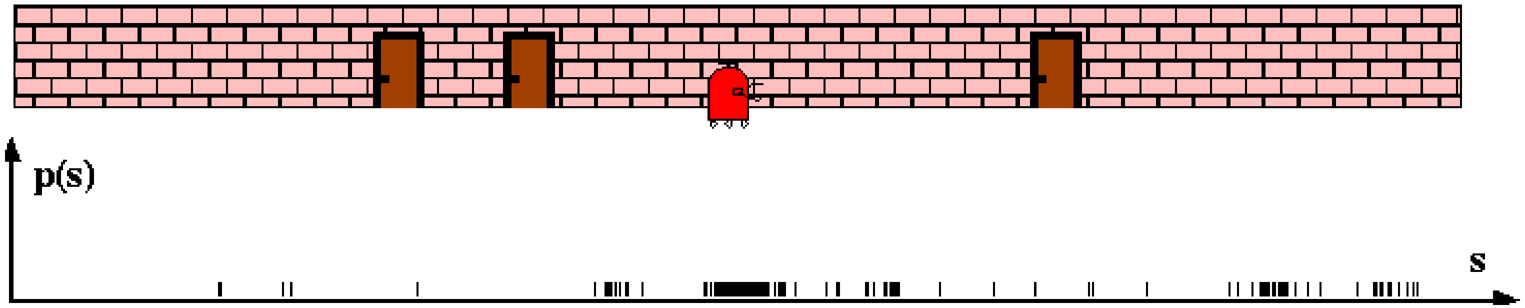


Sensor Information: Importance Sampling

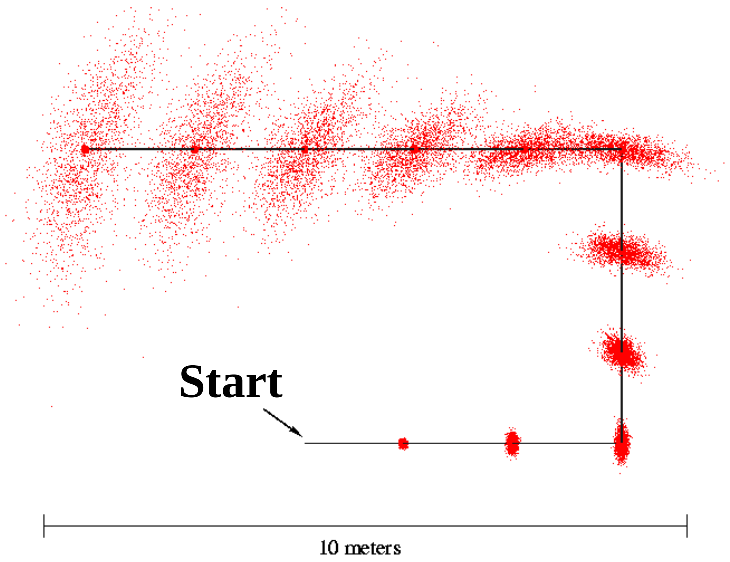


$$Bel^{-}(x) \quad \leftarrow \quad \int p(x\,|\,u, x')\; Bel(x')\; \mathrm{d}\,x'$$



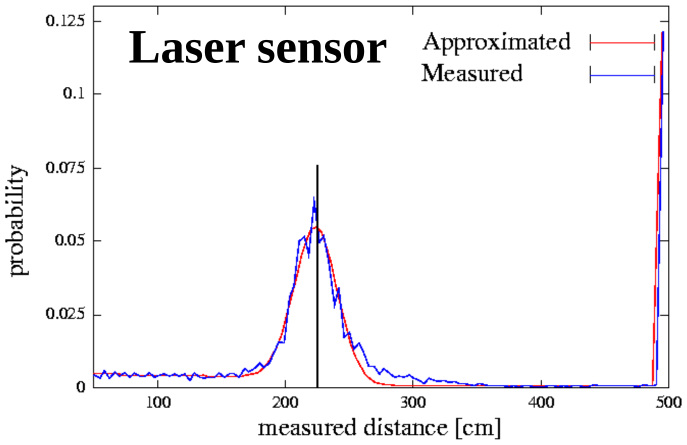
Monte Carlo Localization with Laser

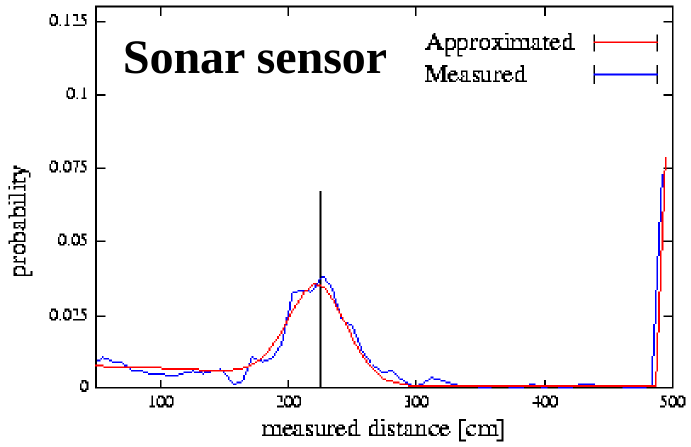
Stochastic motion model



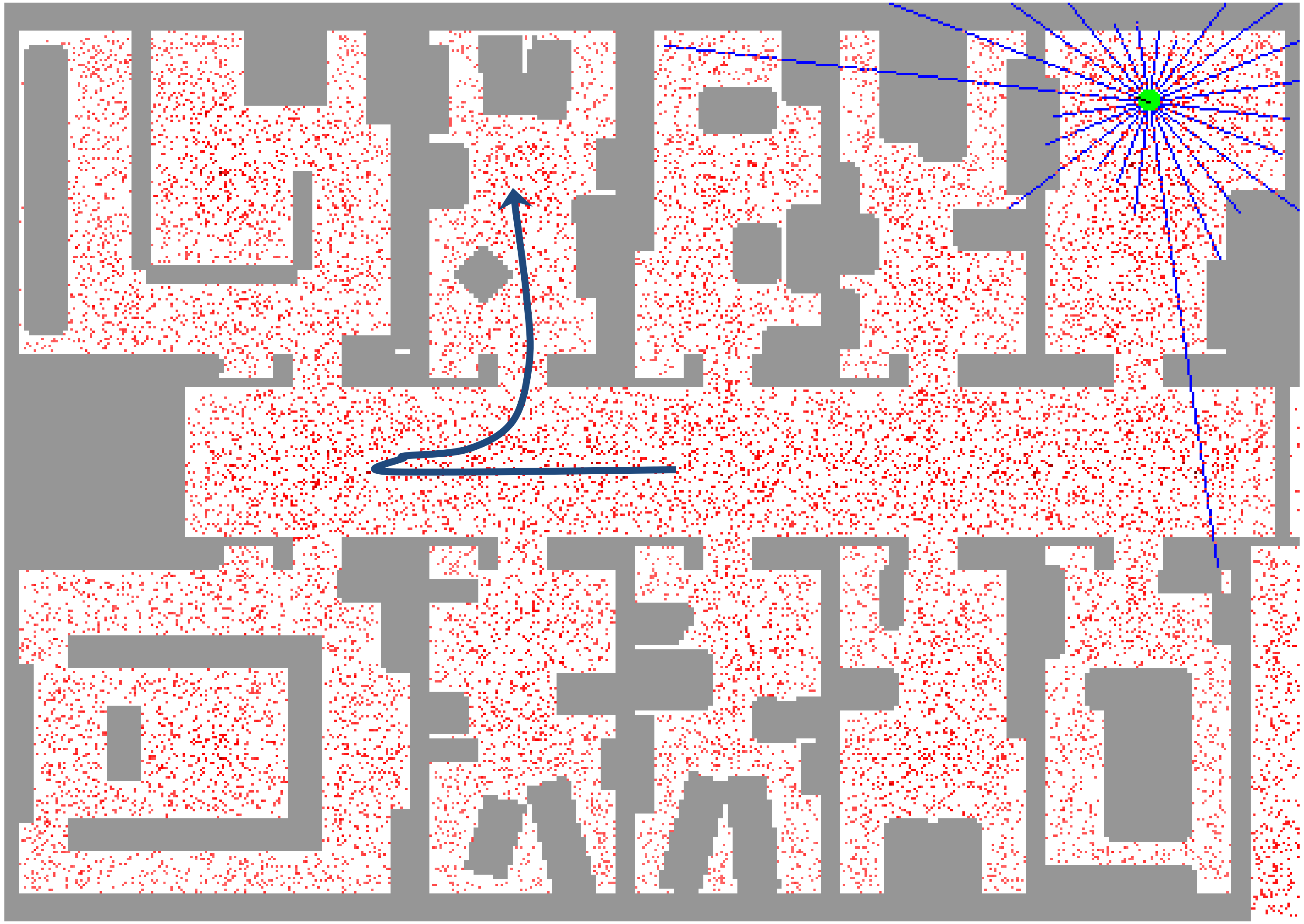
Start
10 meters

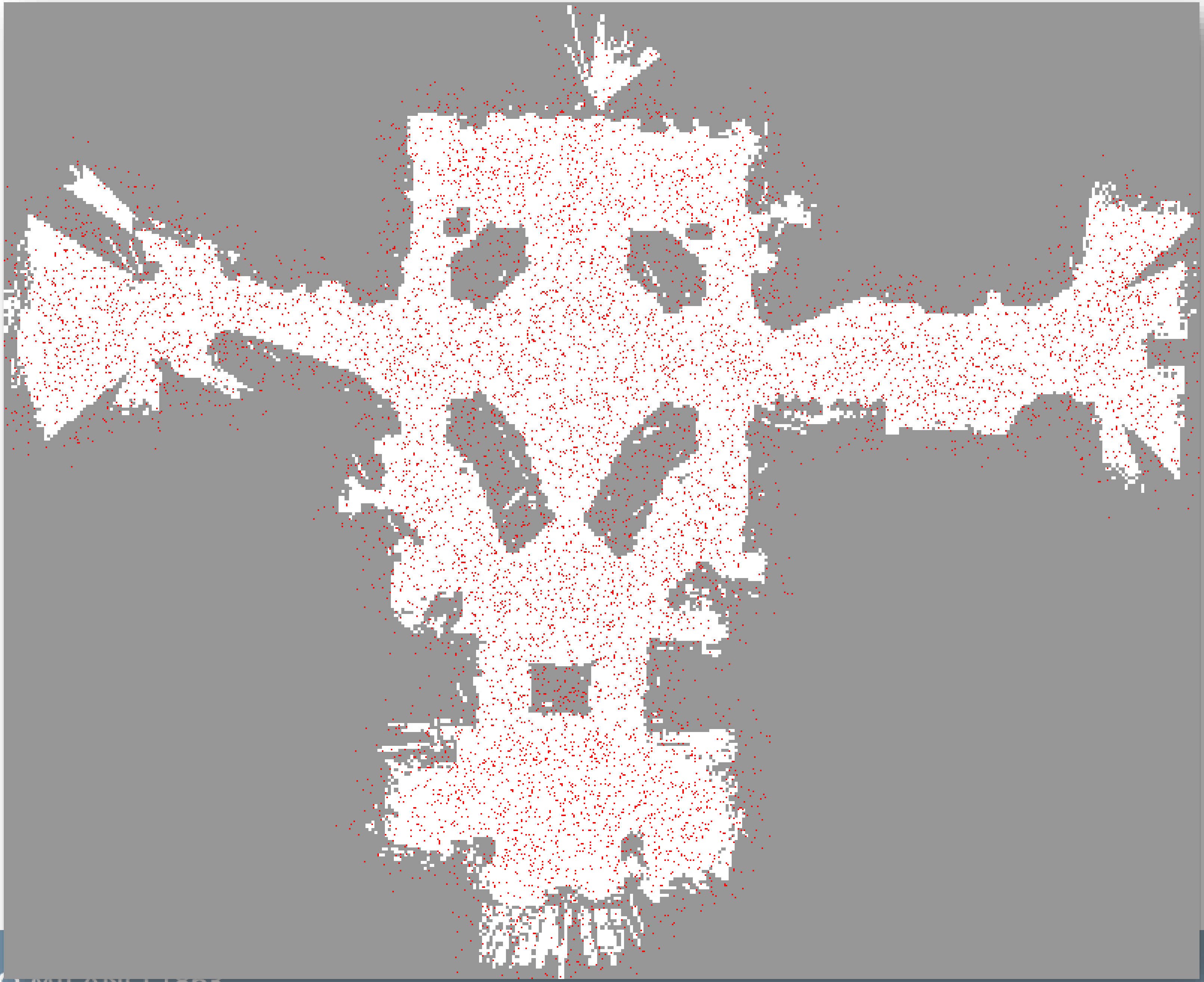
Range sensor model

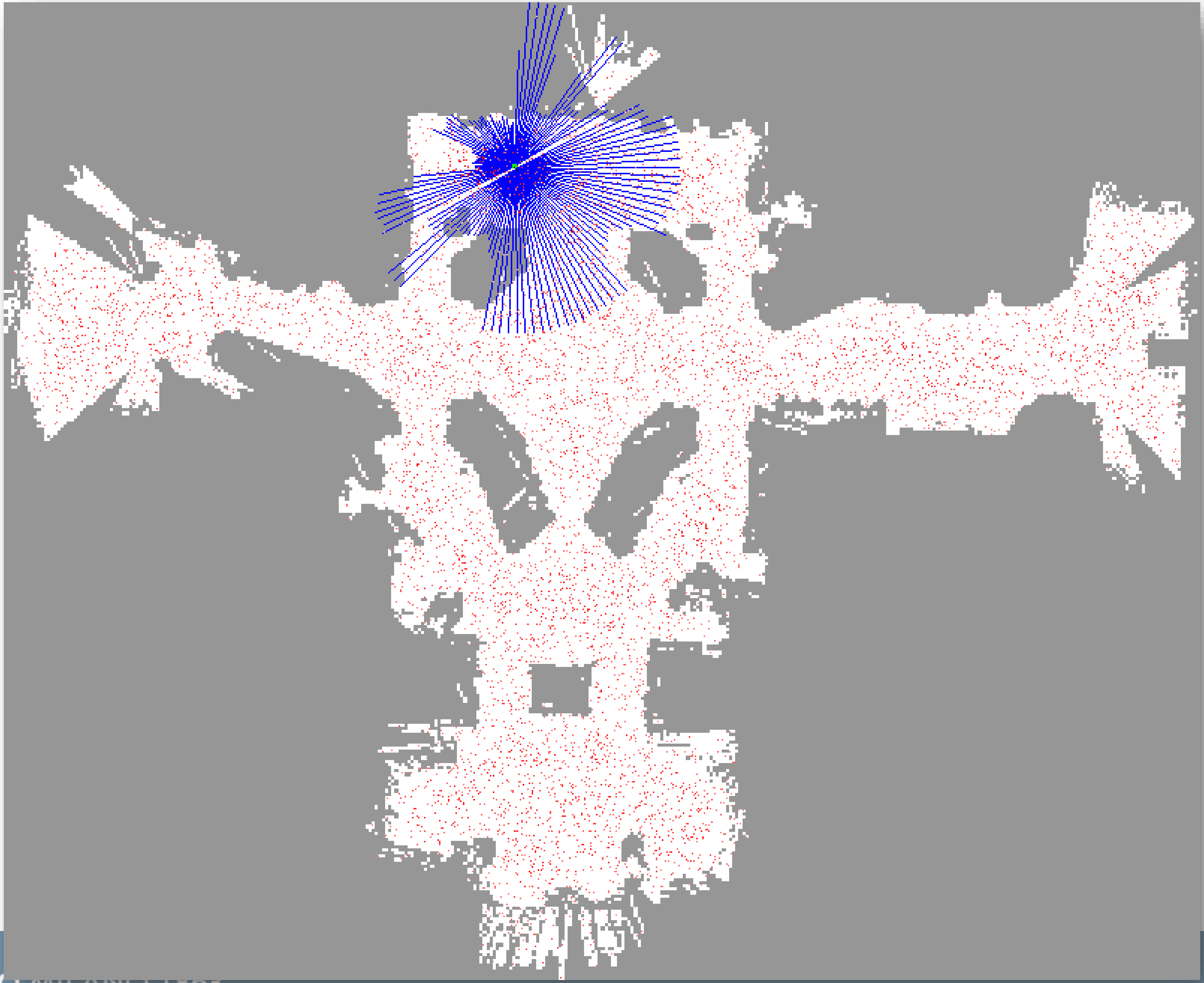


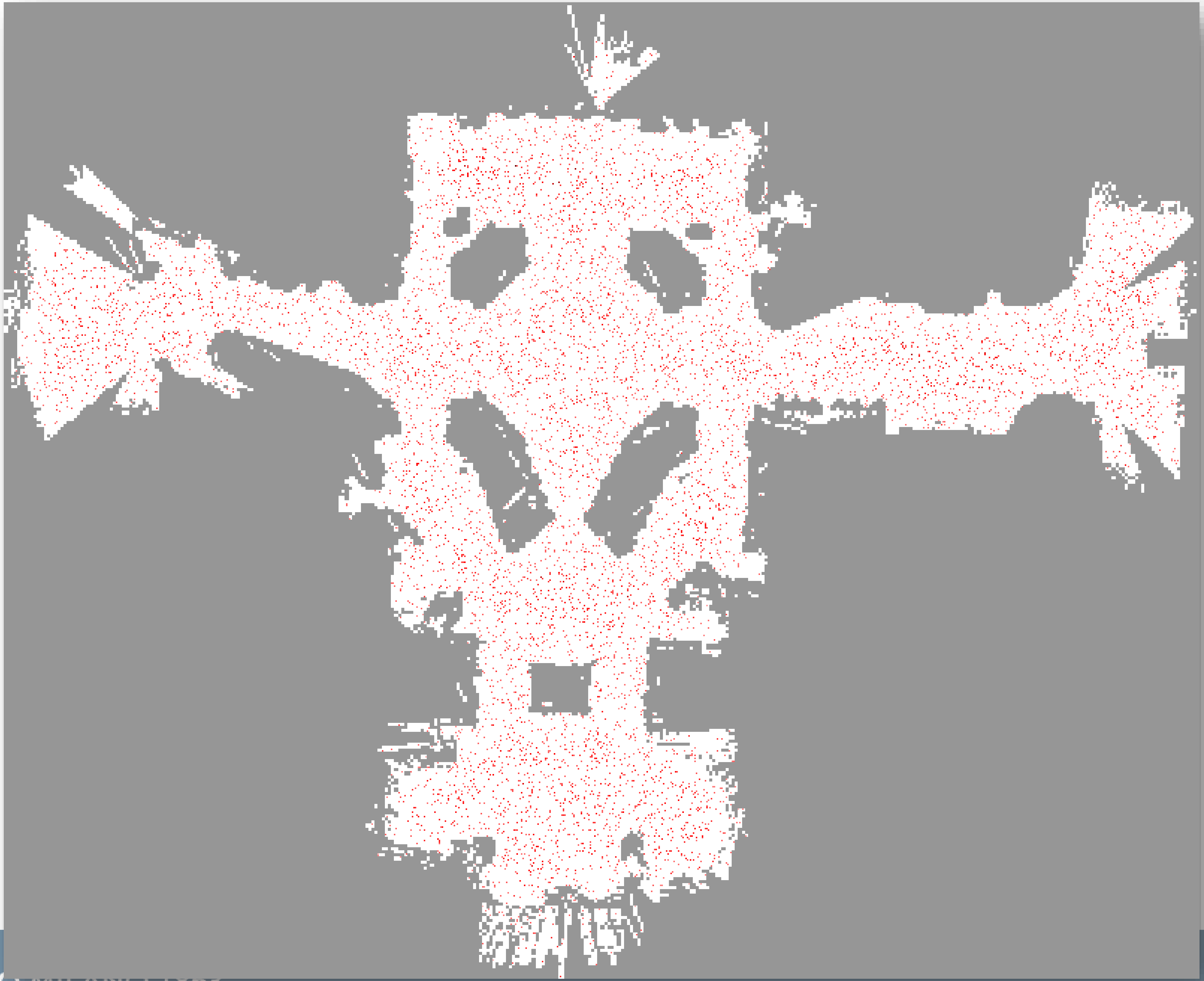


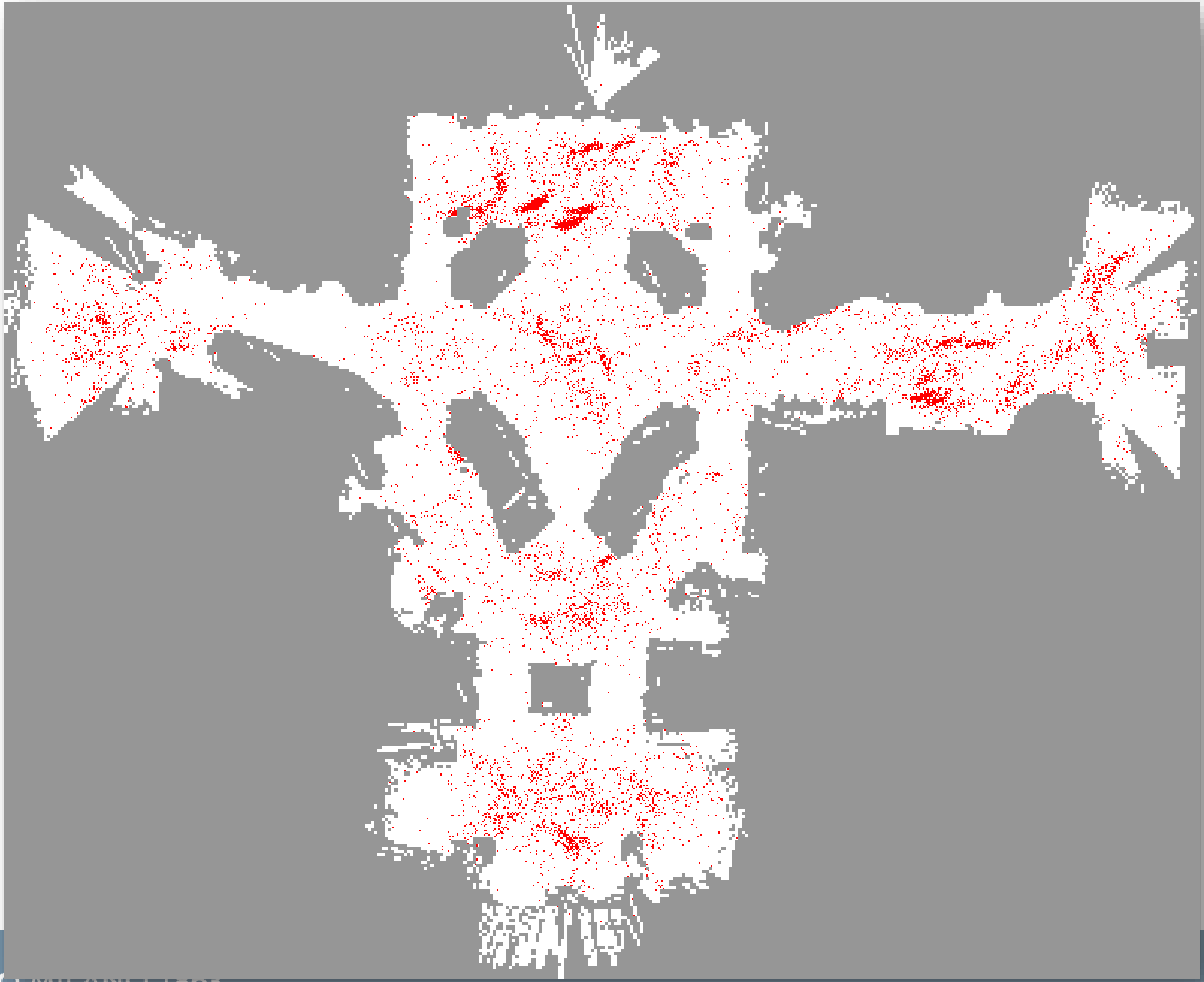
Sample-based Localization (sonar)

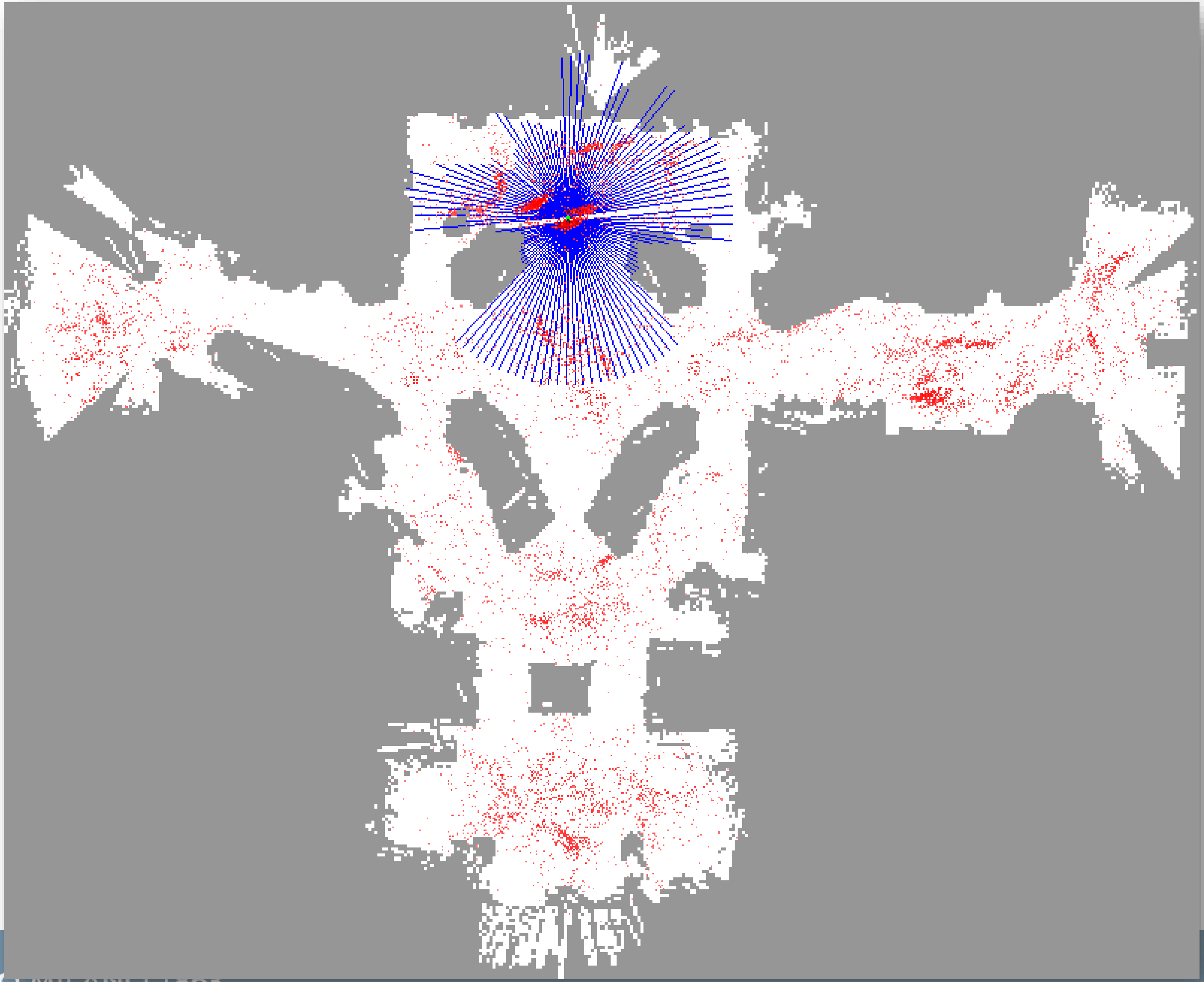


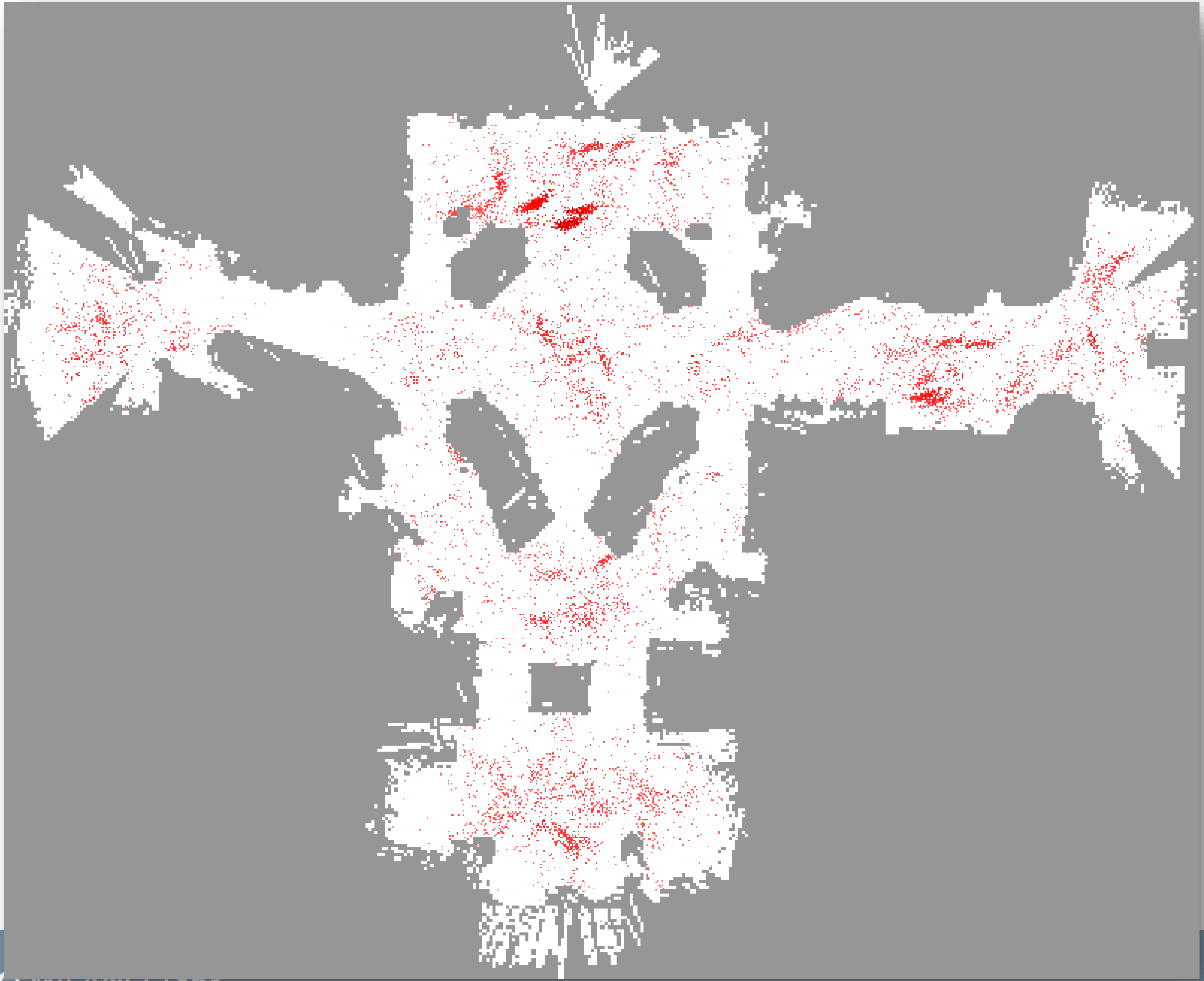


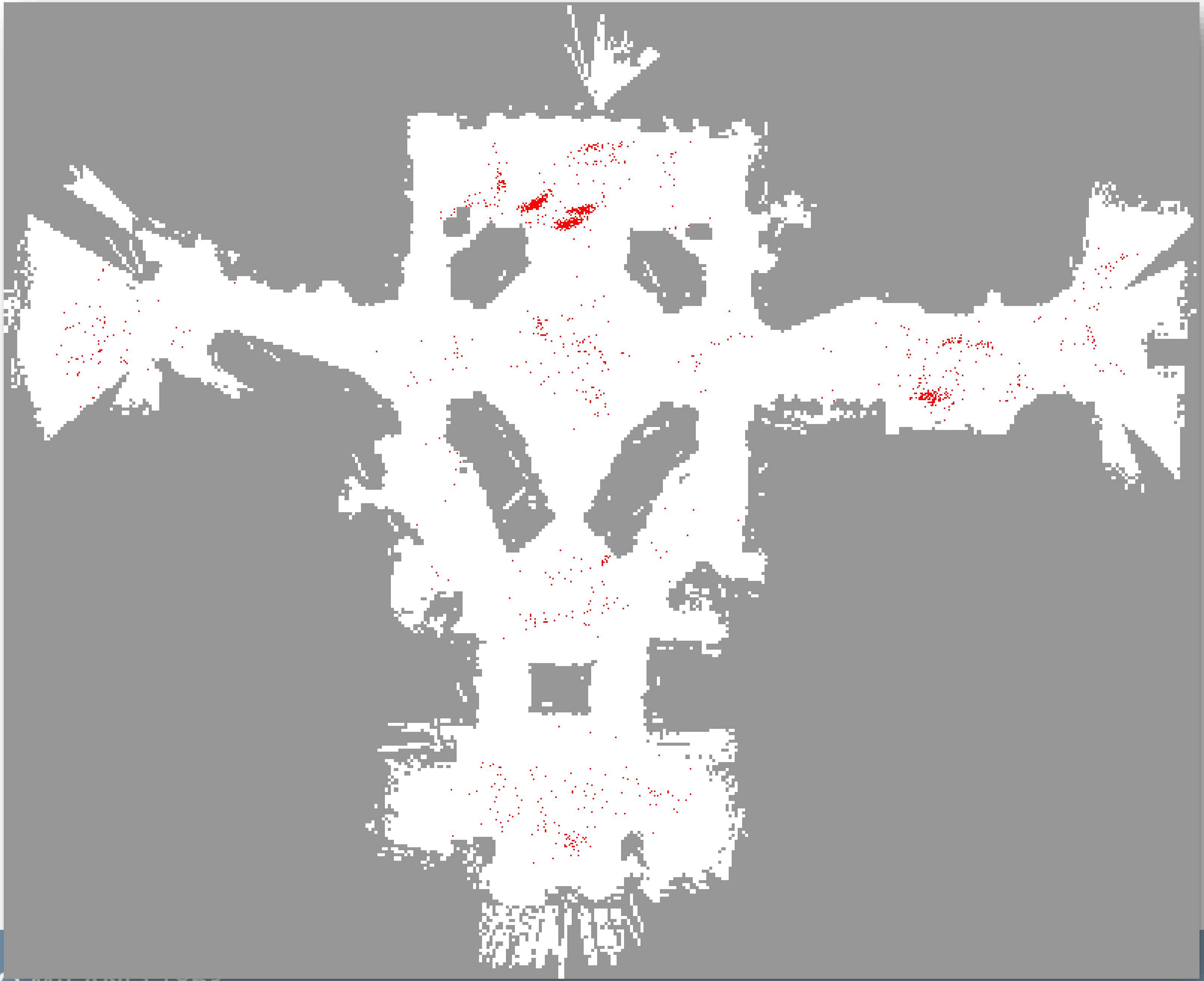


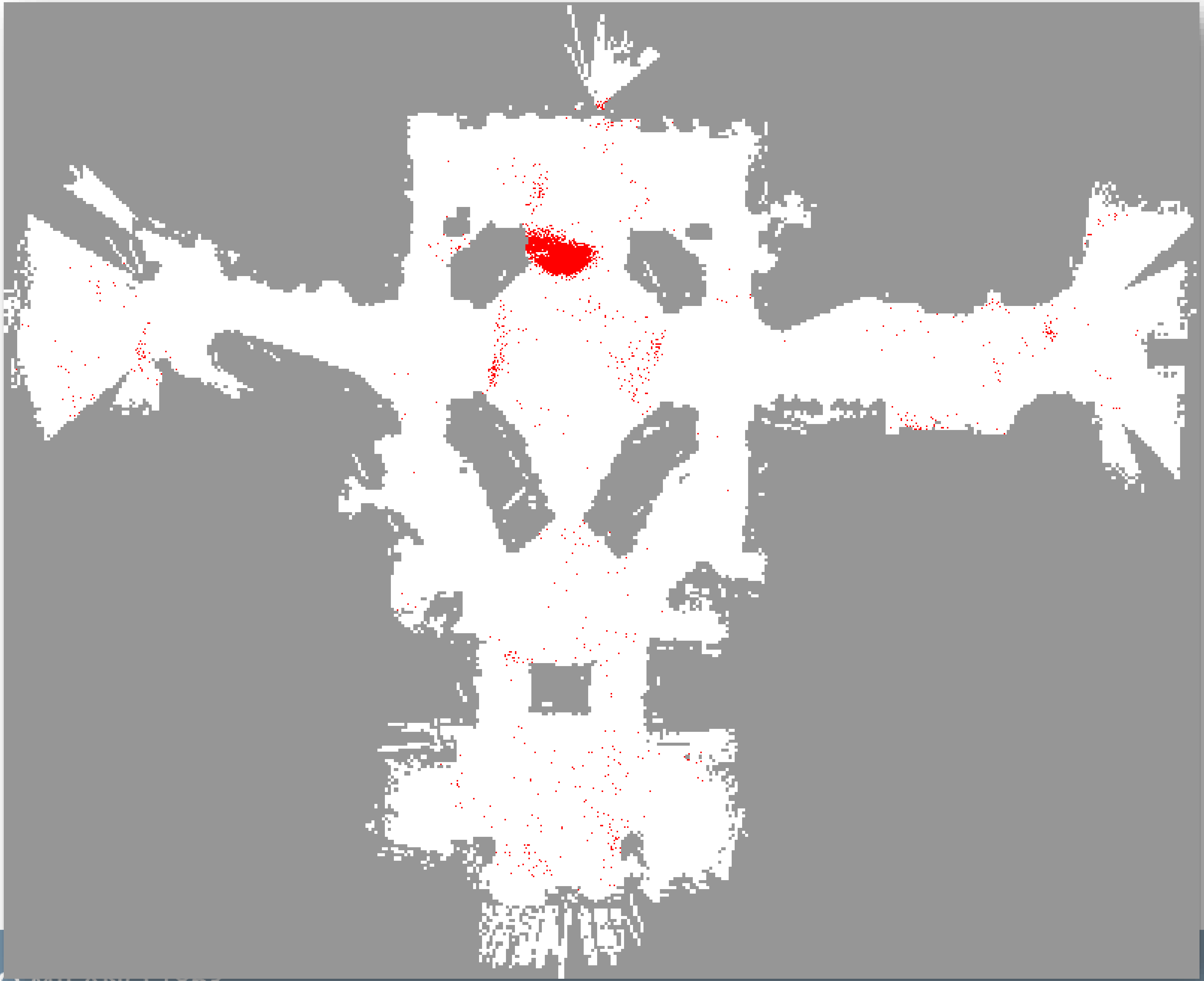


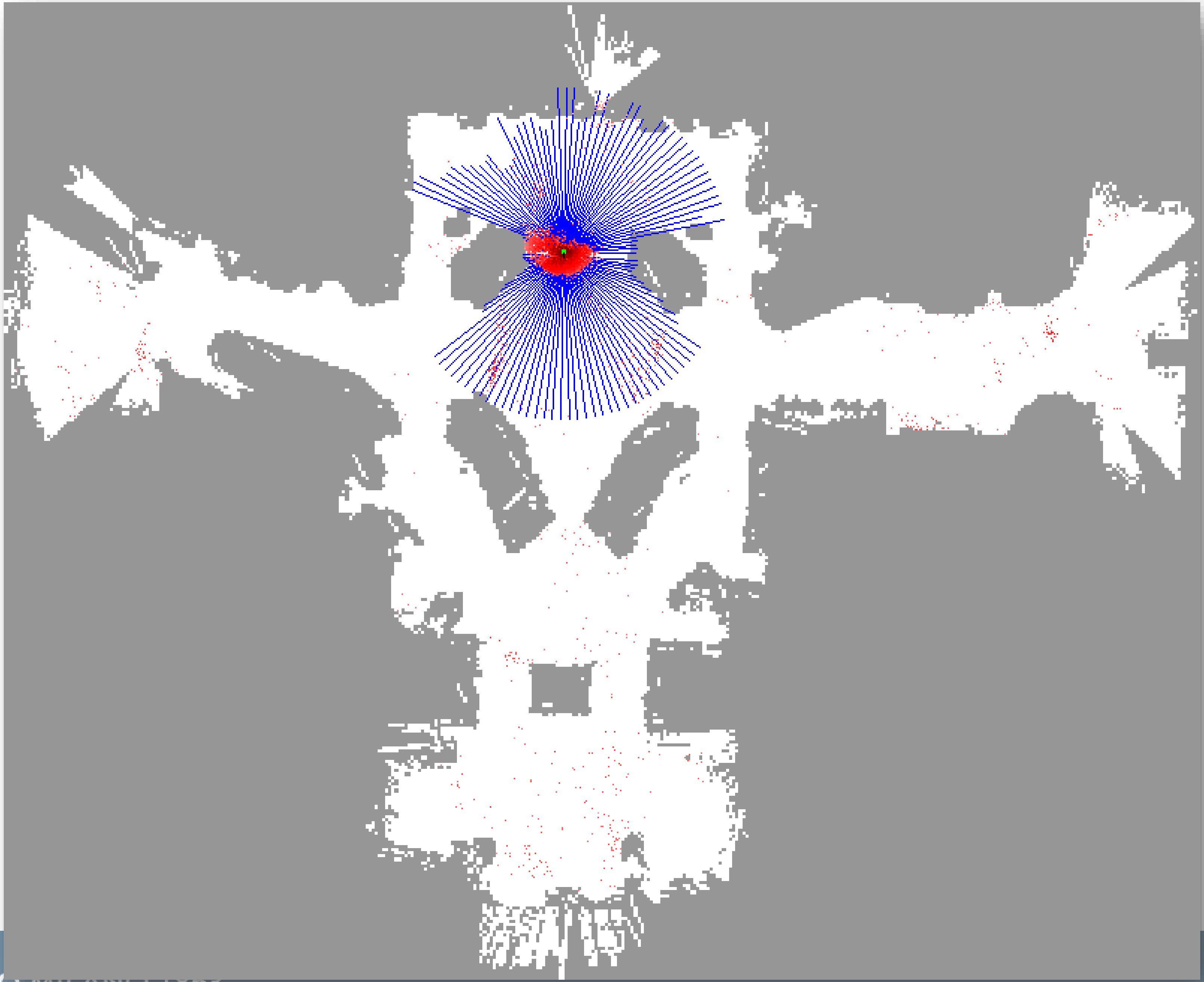


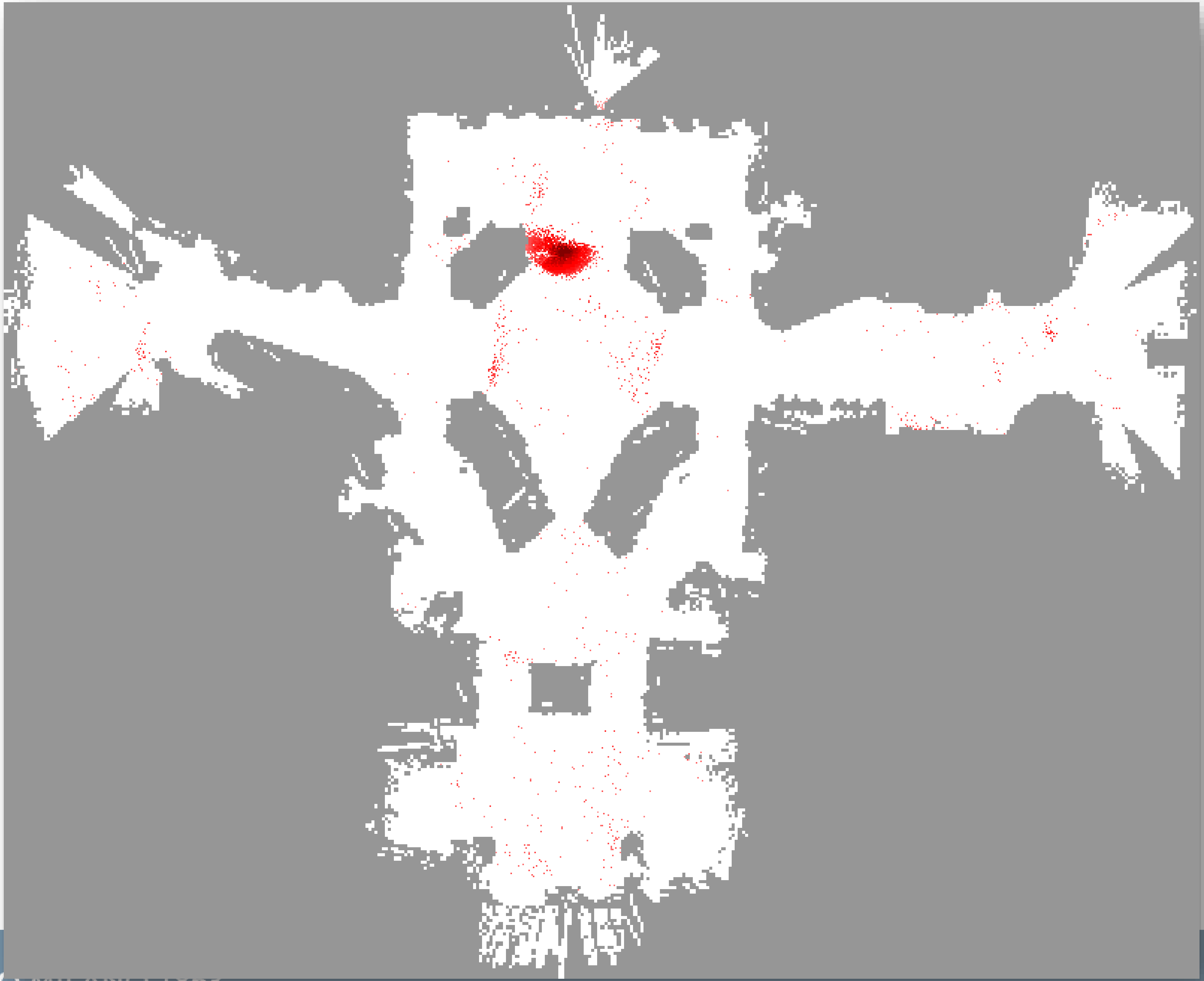


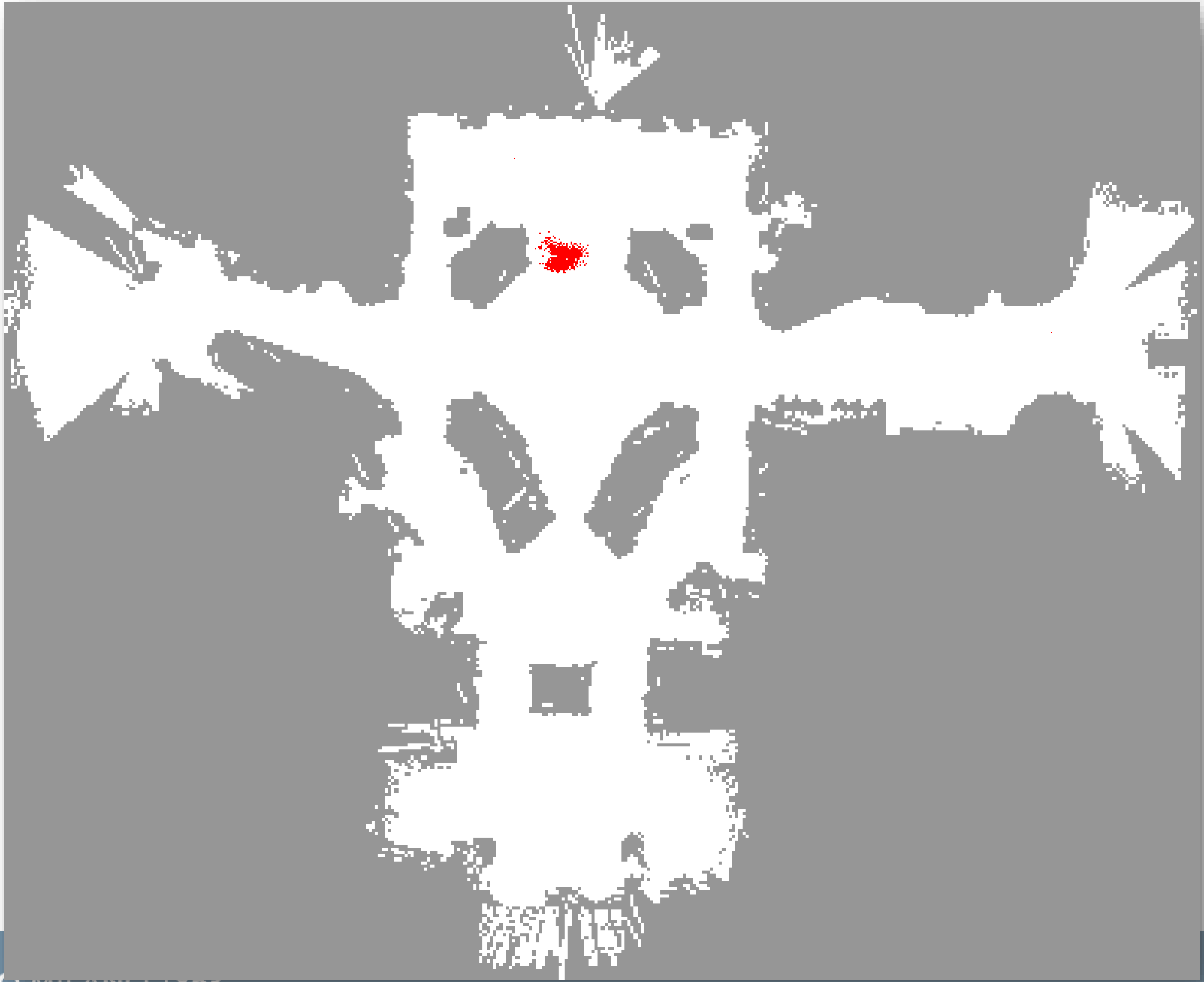


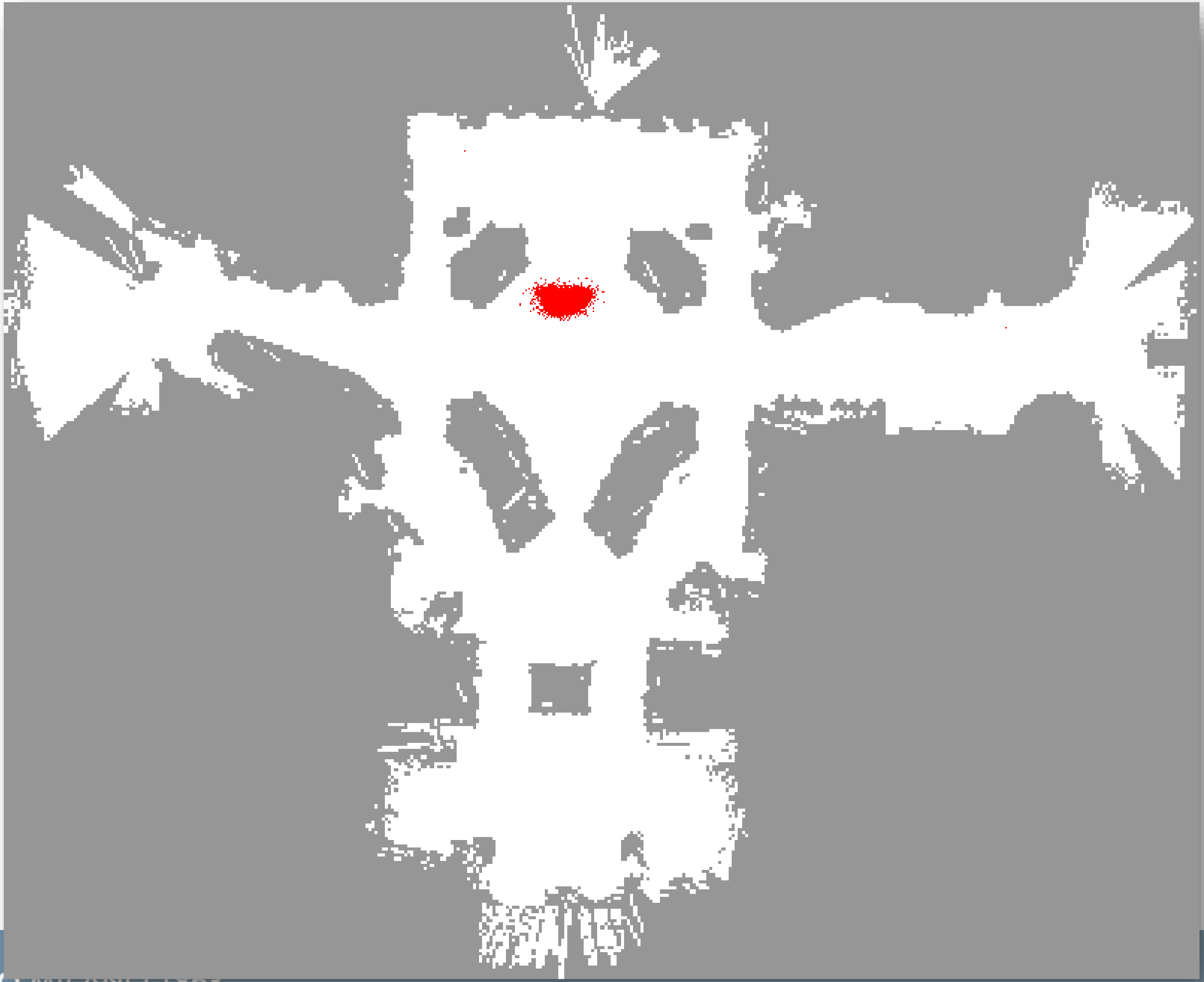


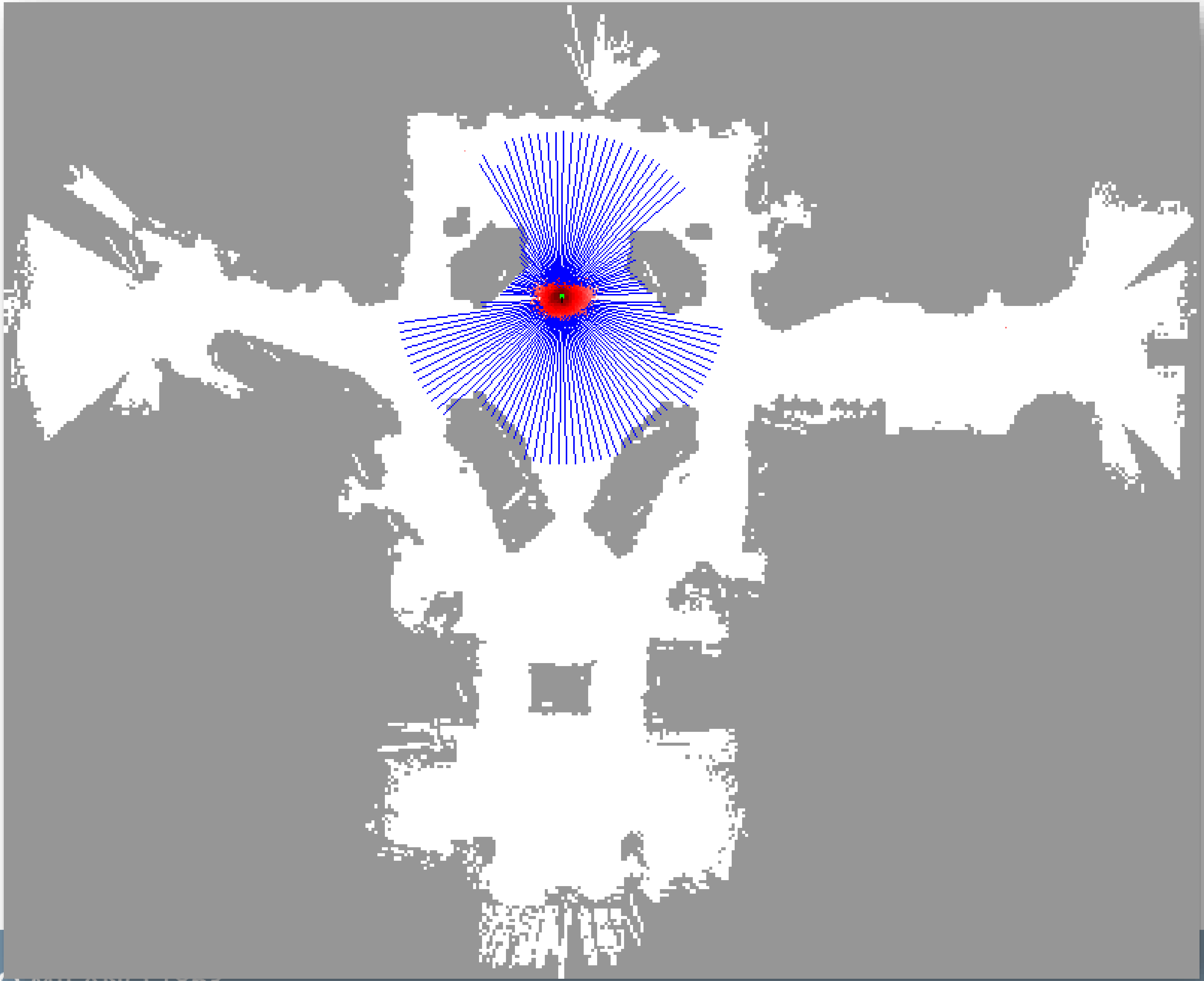


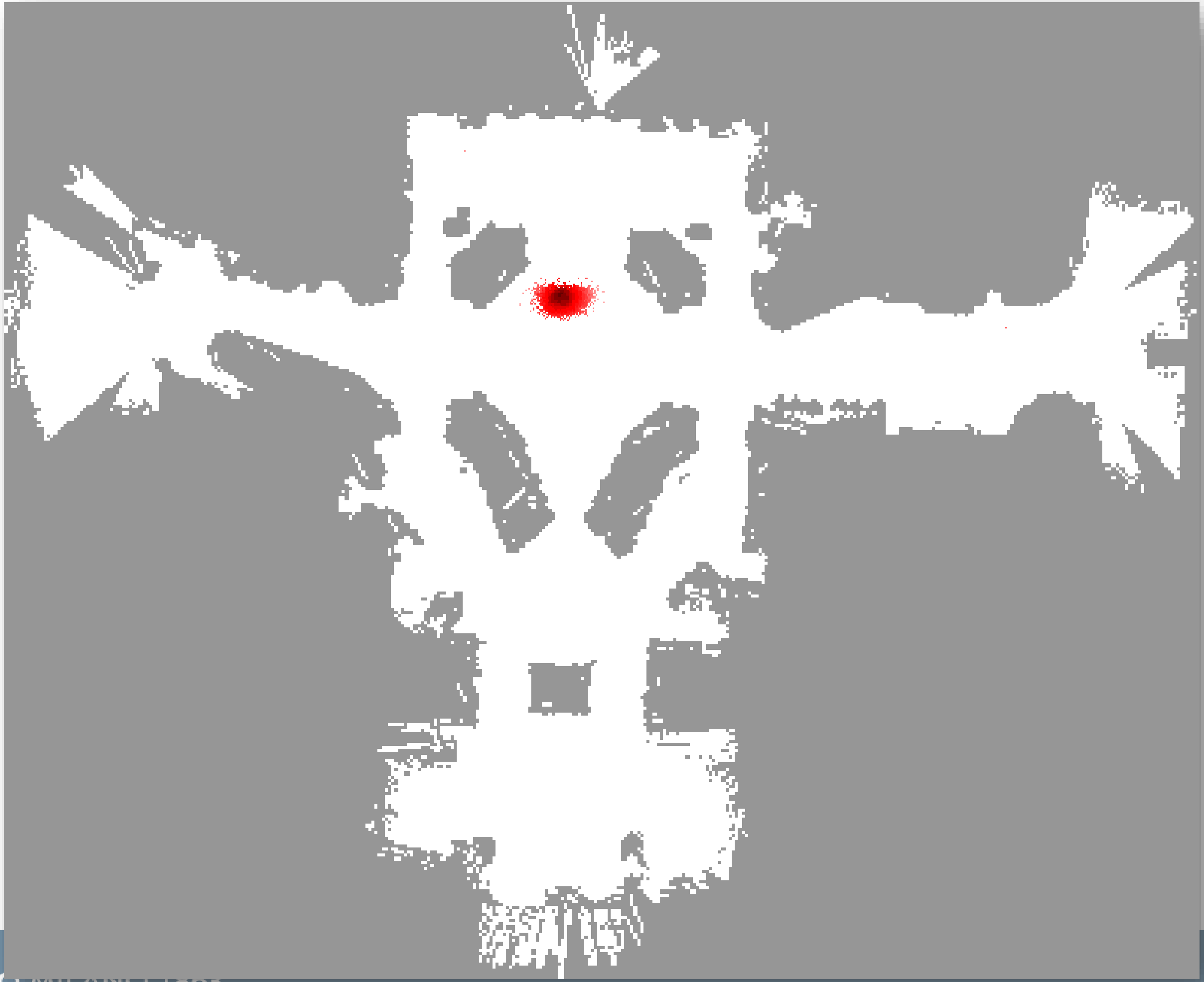


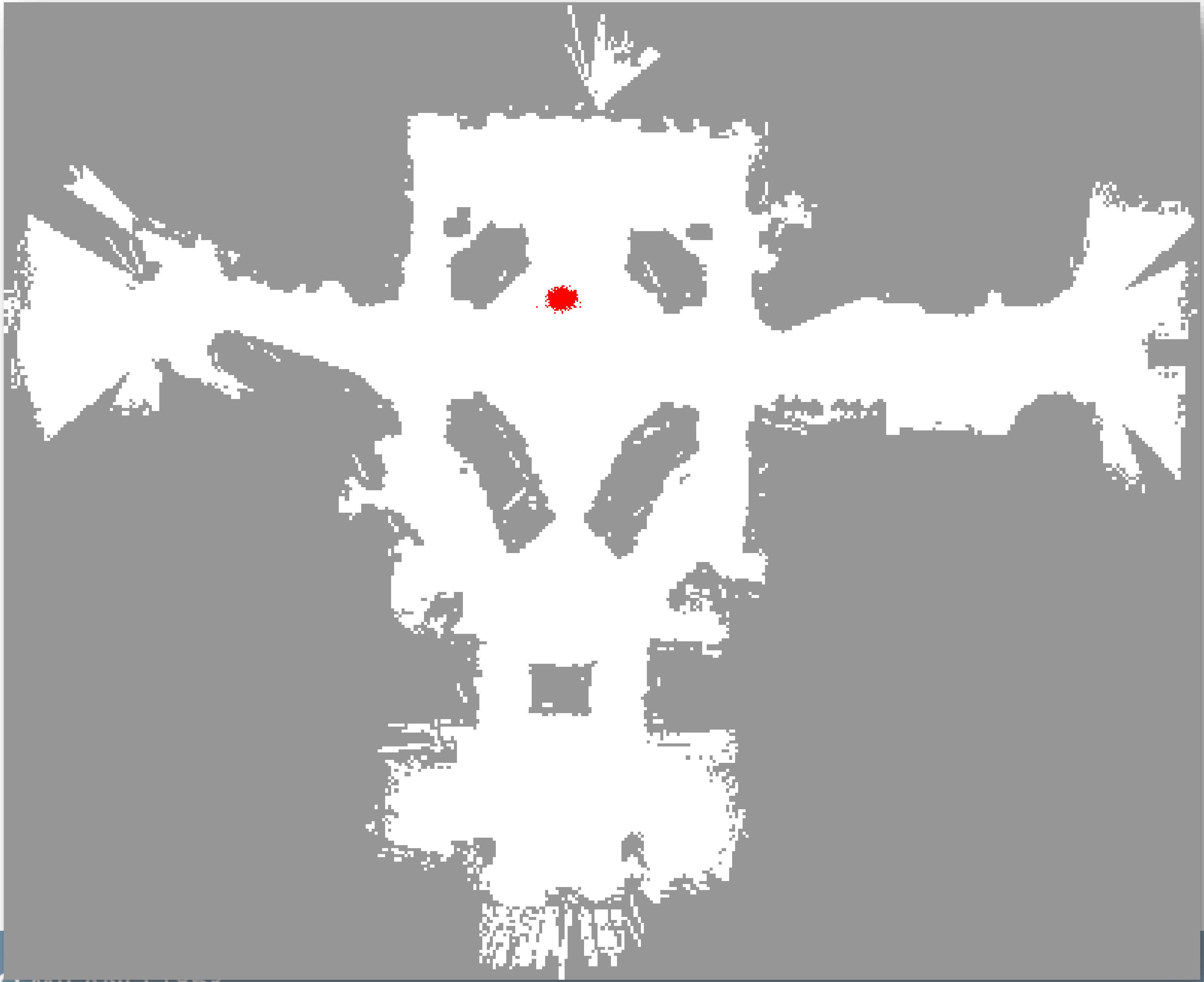


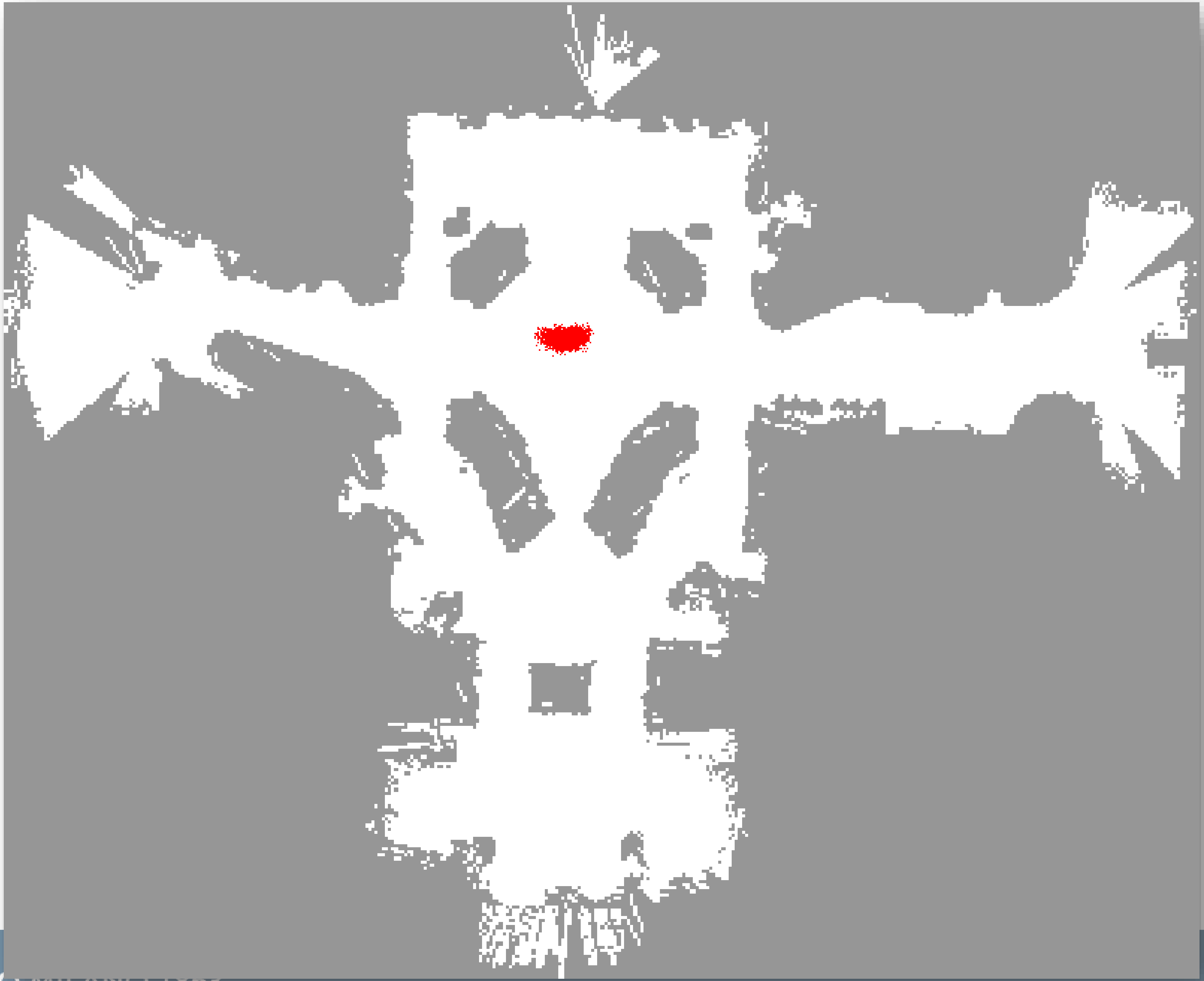


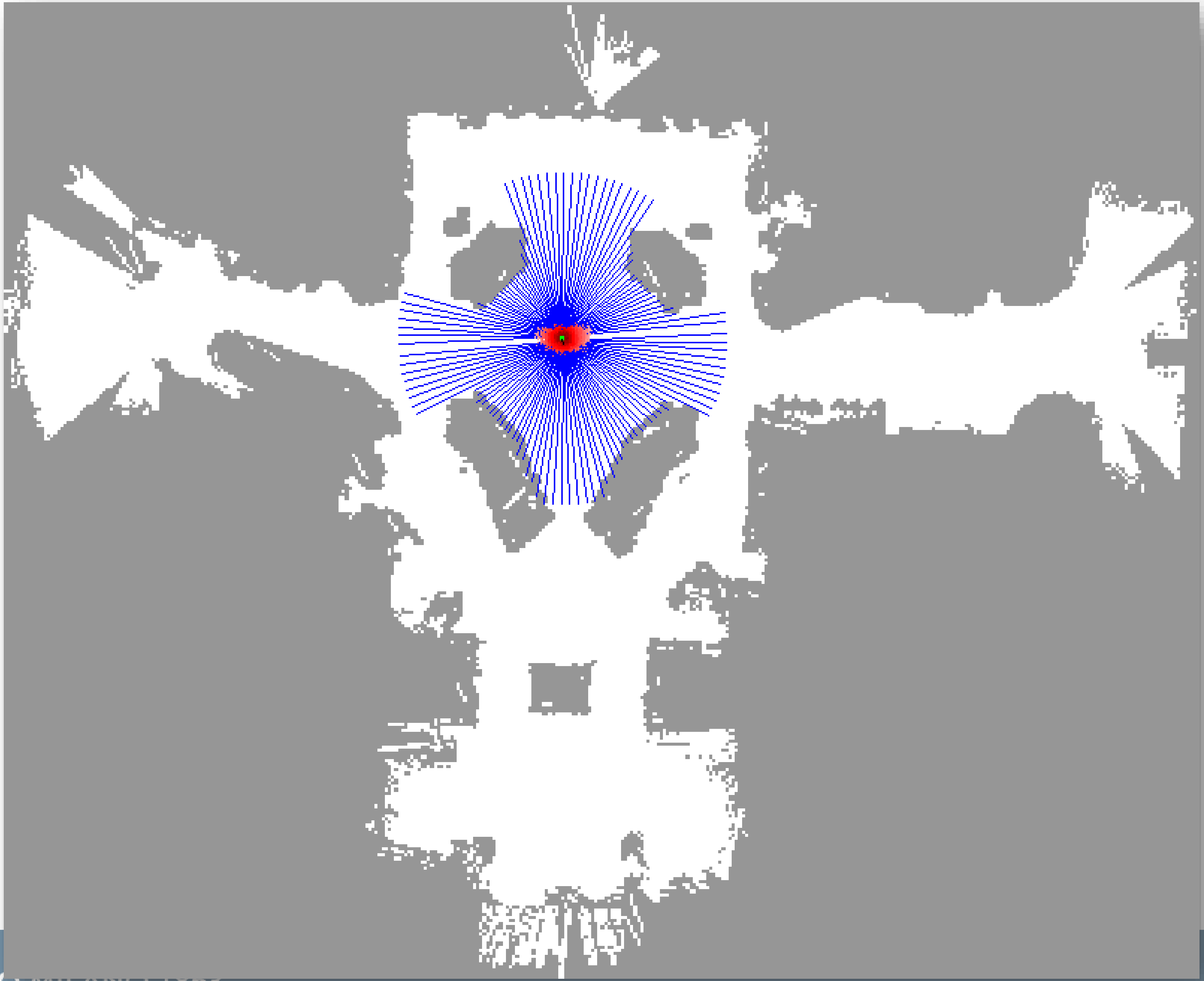


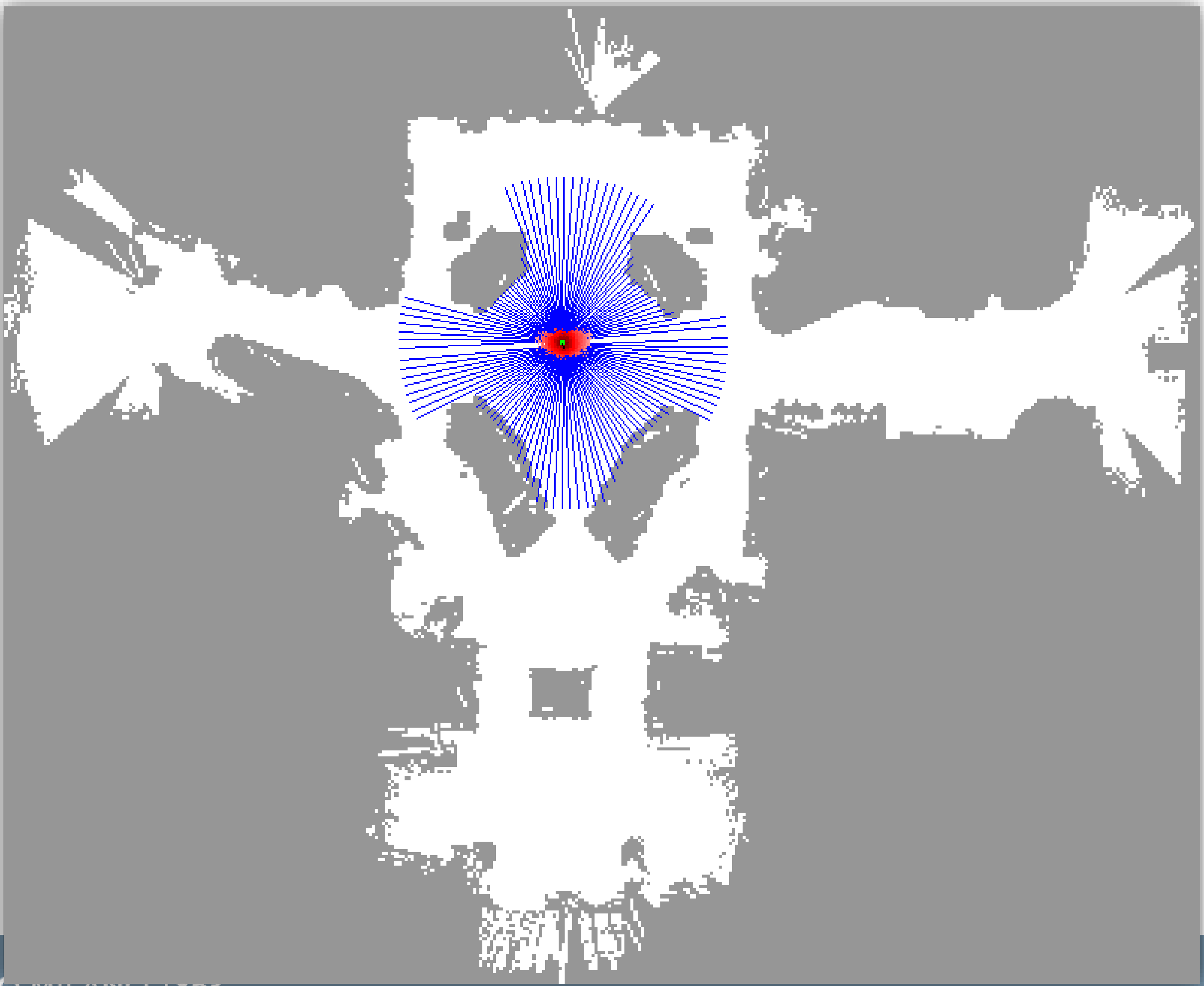




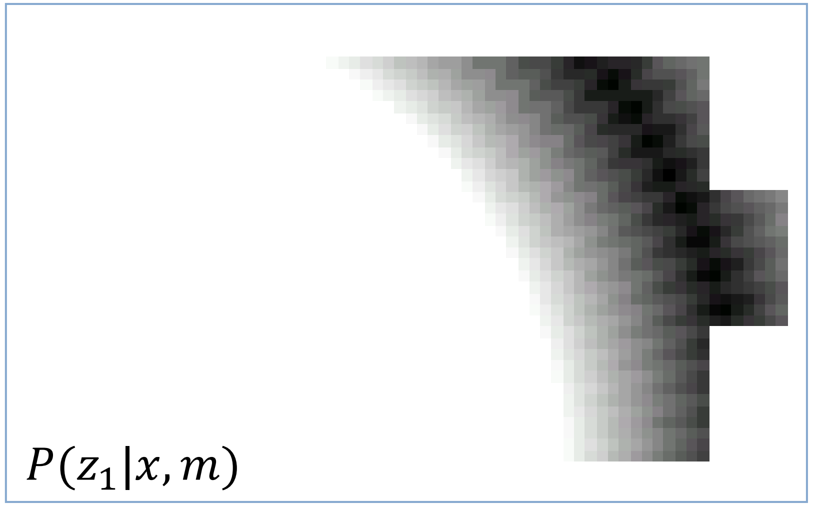


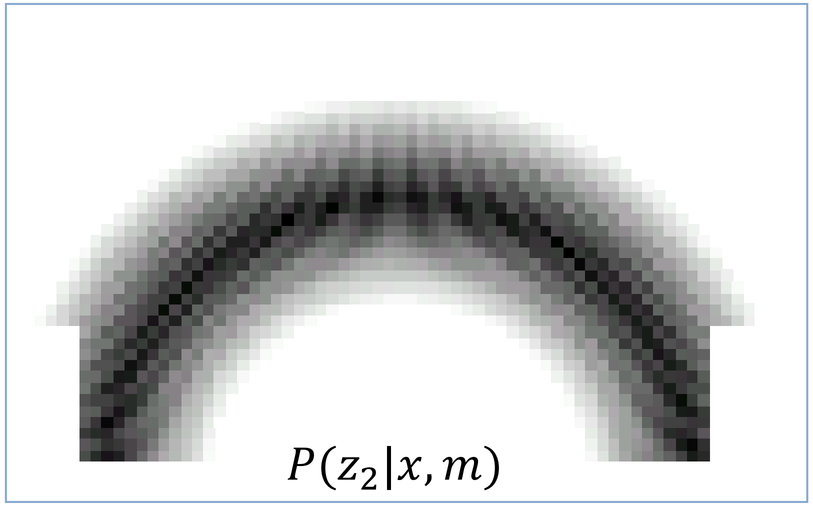


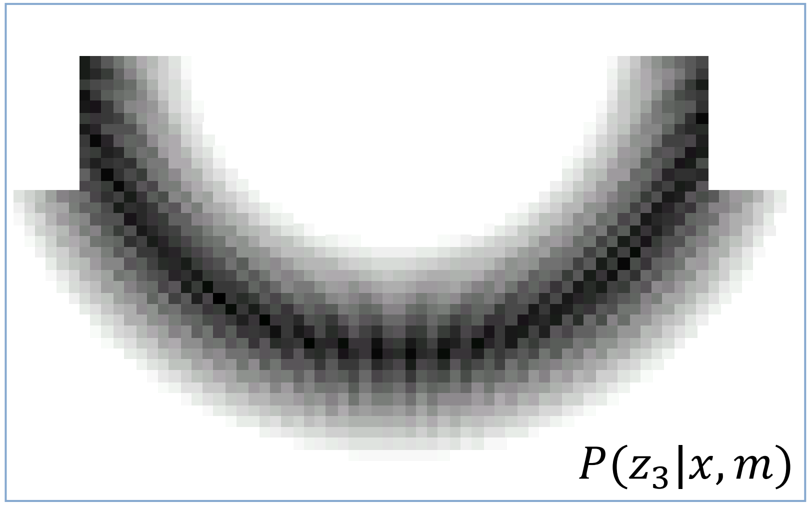


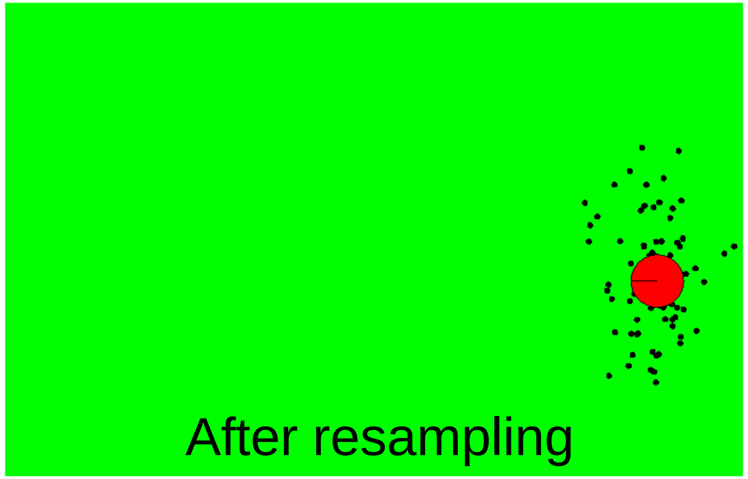


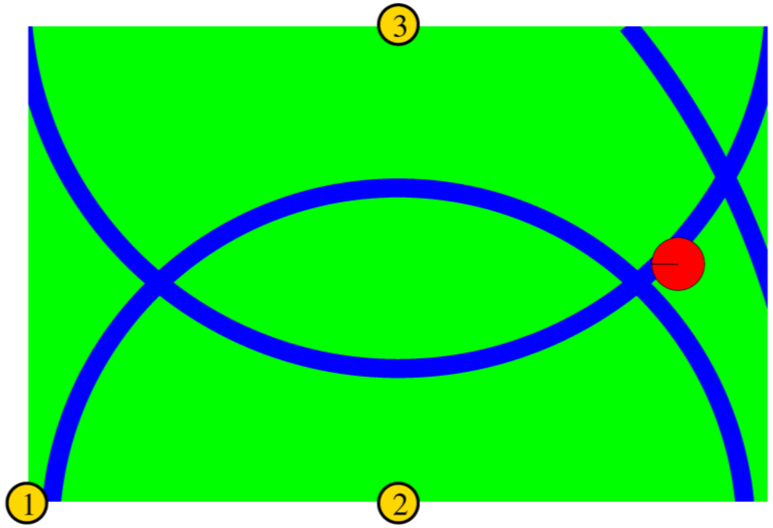
RoboCup Example



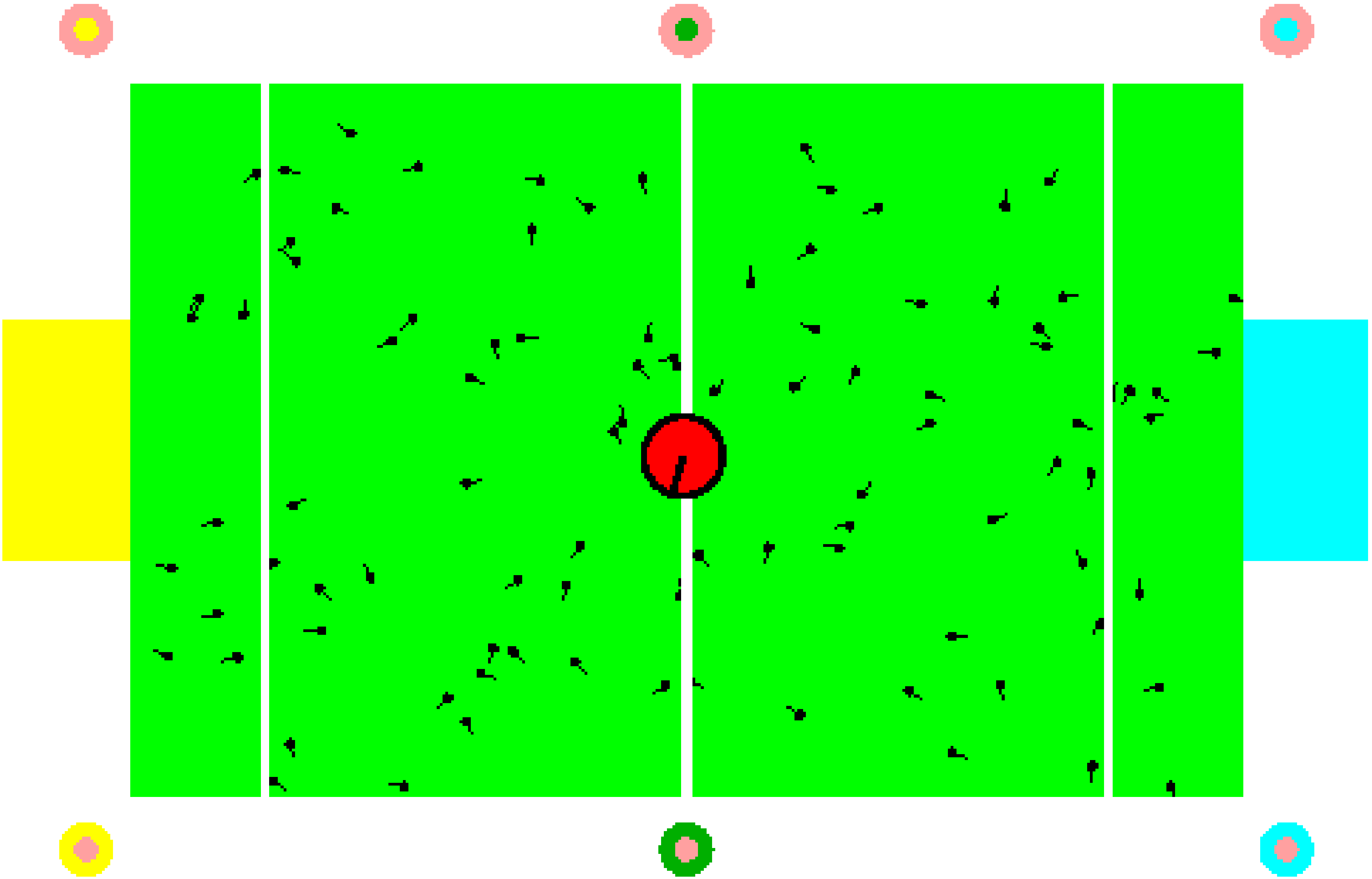





After resampling



Localization for AIBO robots



Project Minerva

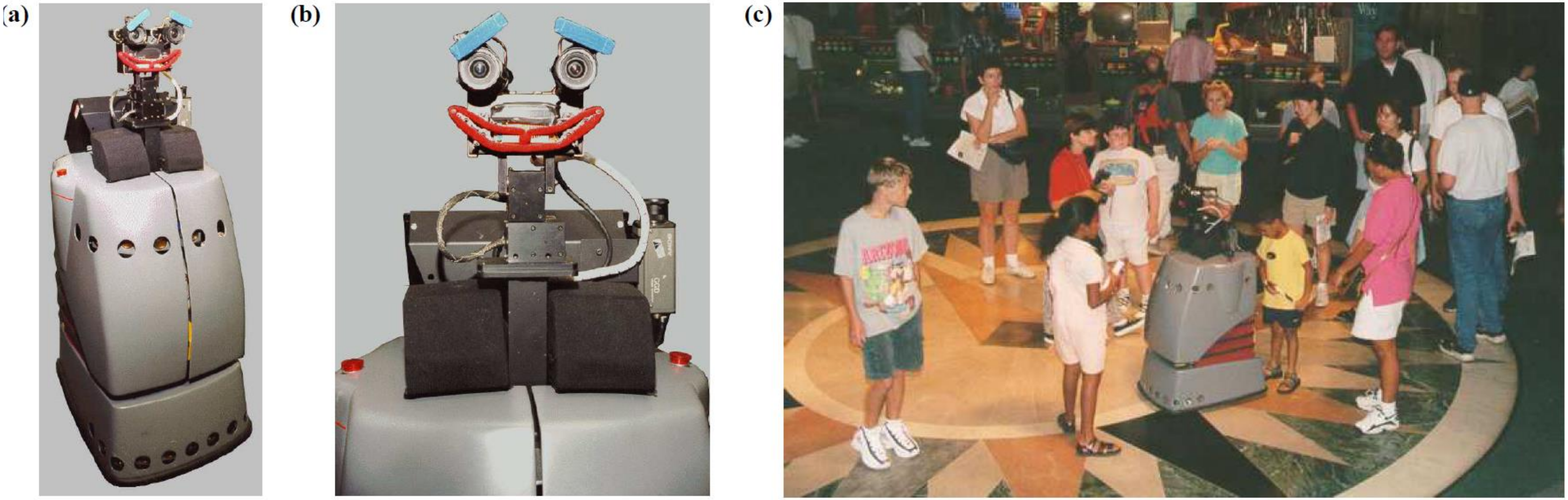
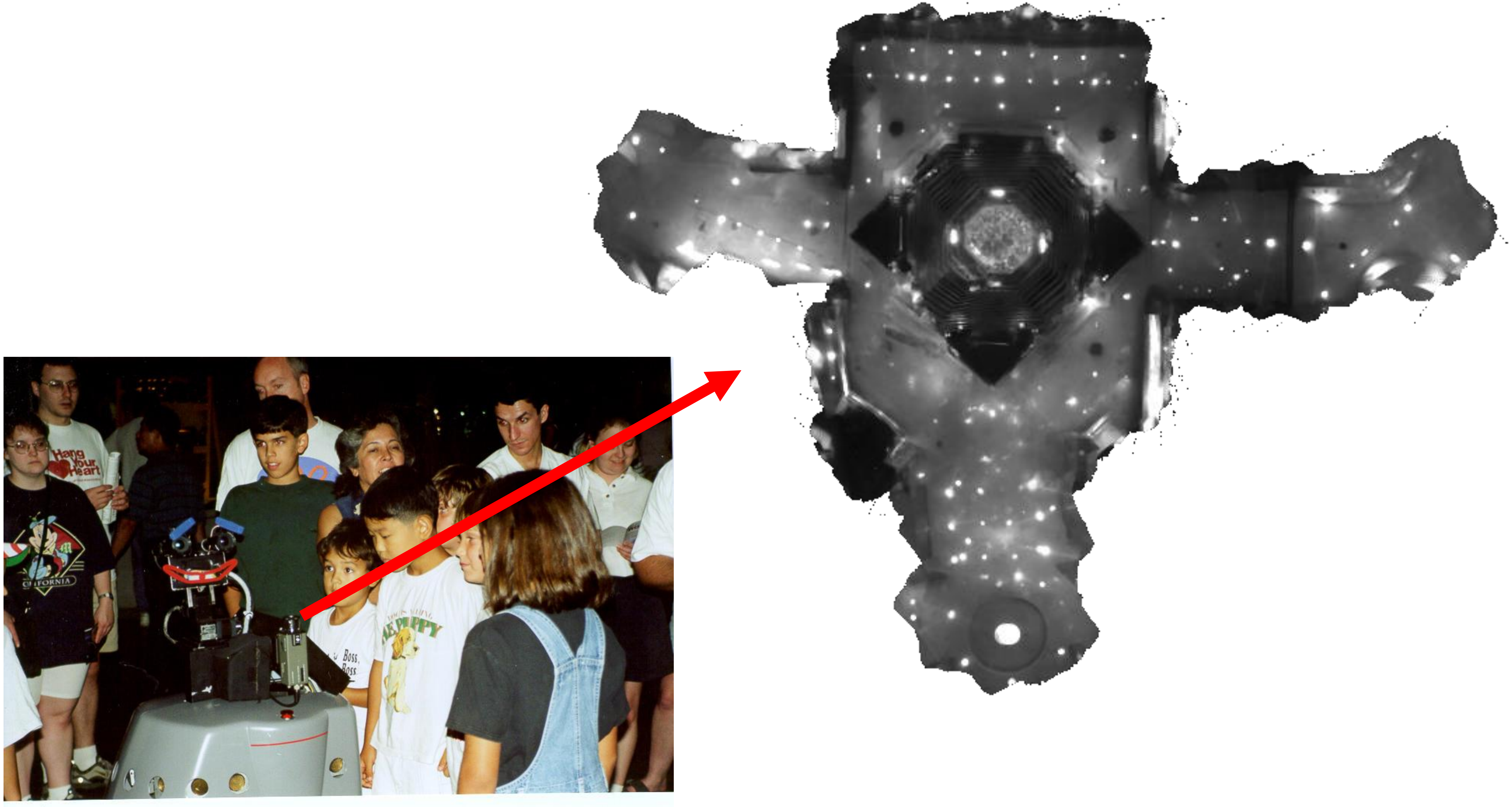
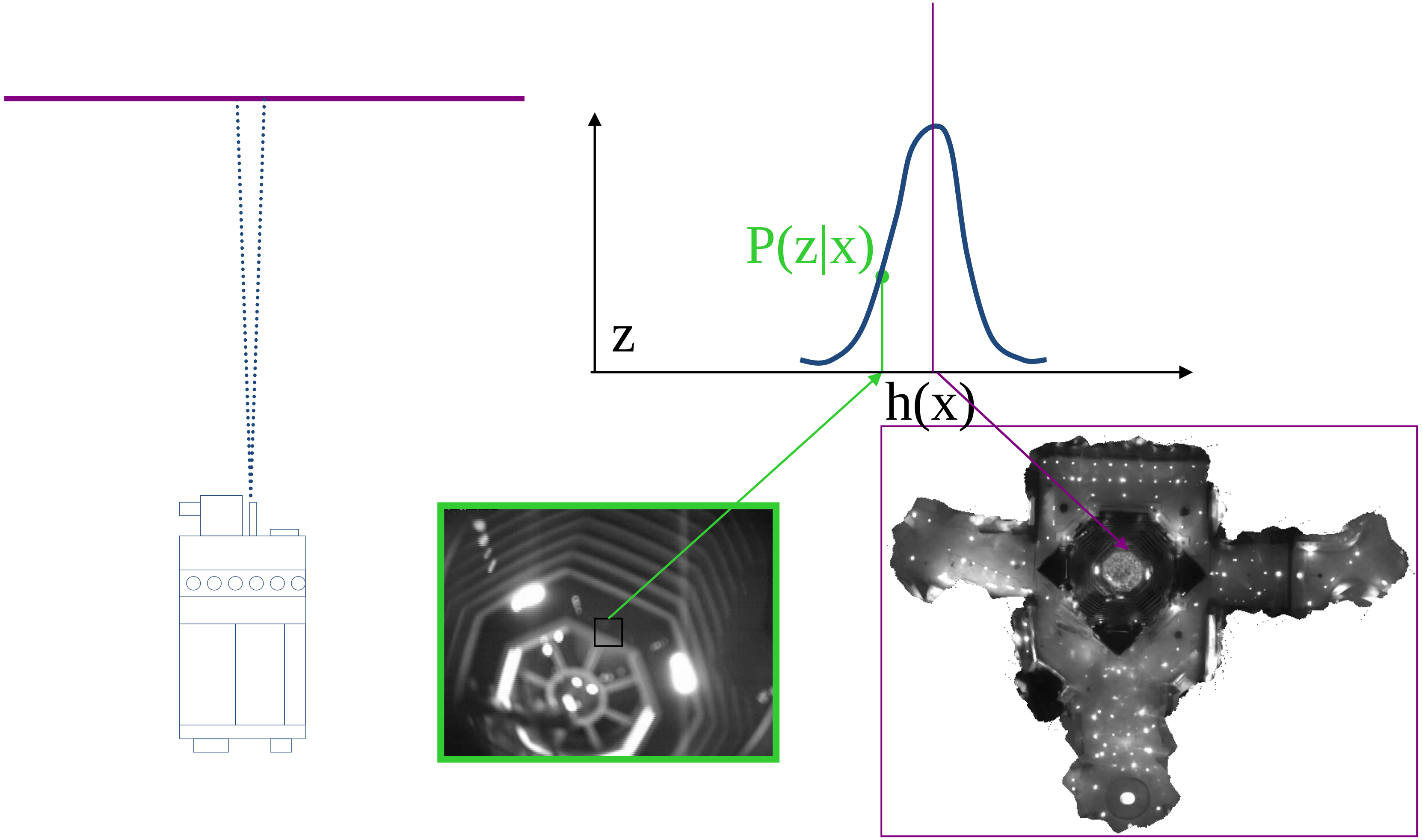


Figure 1: (a) Minerva. (b) Minerva's motorized face. (c) Minerva gives a tour in the Smithsonian's National Museum of American History.

Using Ceiling Maps for Localization

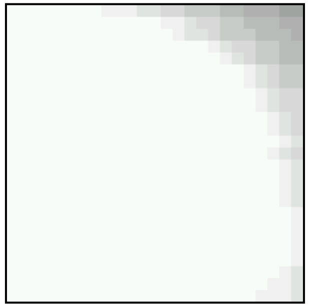


Vision-based Localization

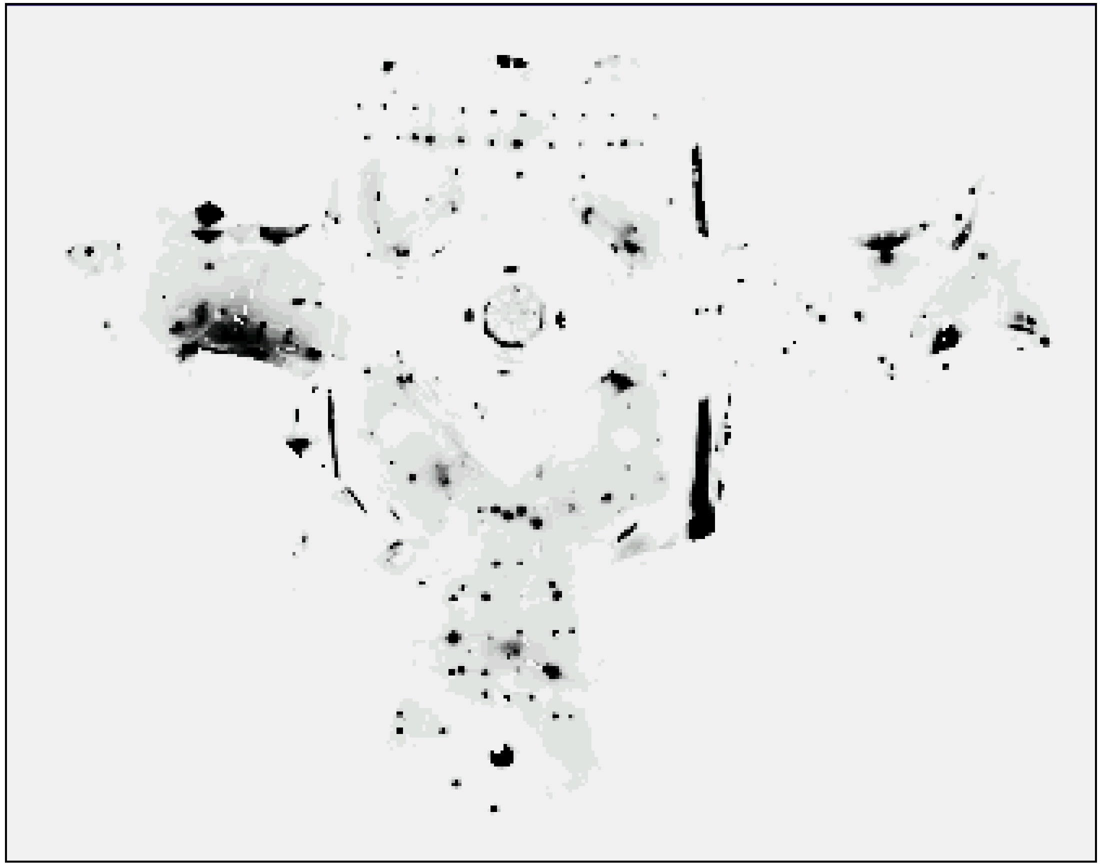


Under a Light

Measurement z:

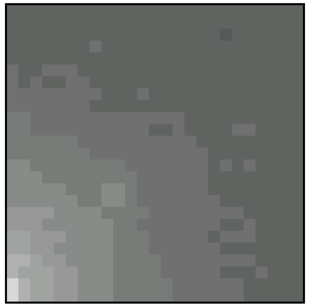


***P(z|x)*:**

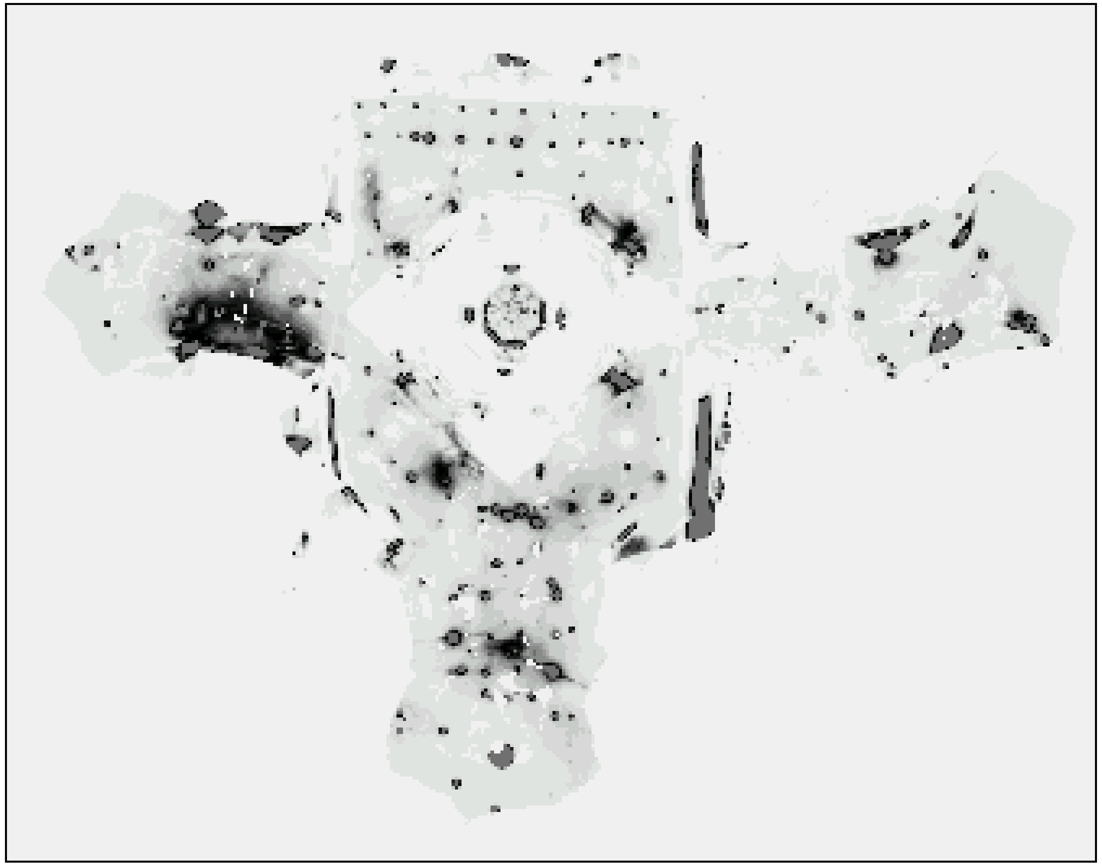


Next to a Light

Measurement z:

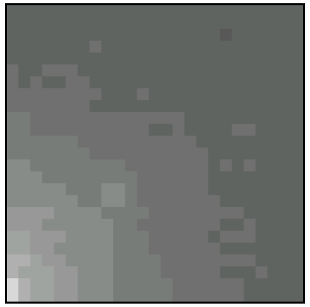


***P(z|x)*:**

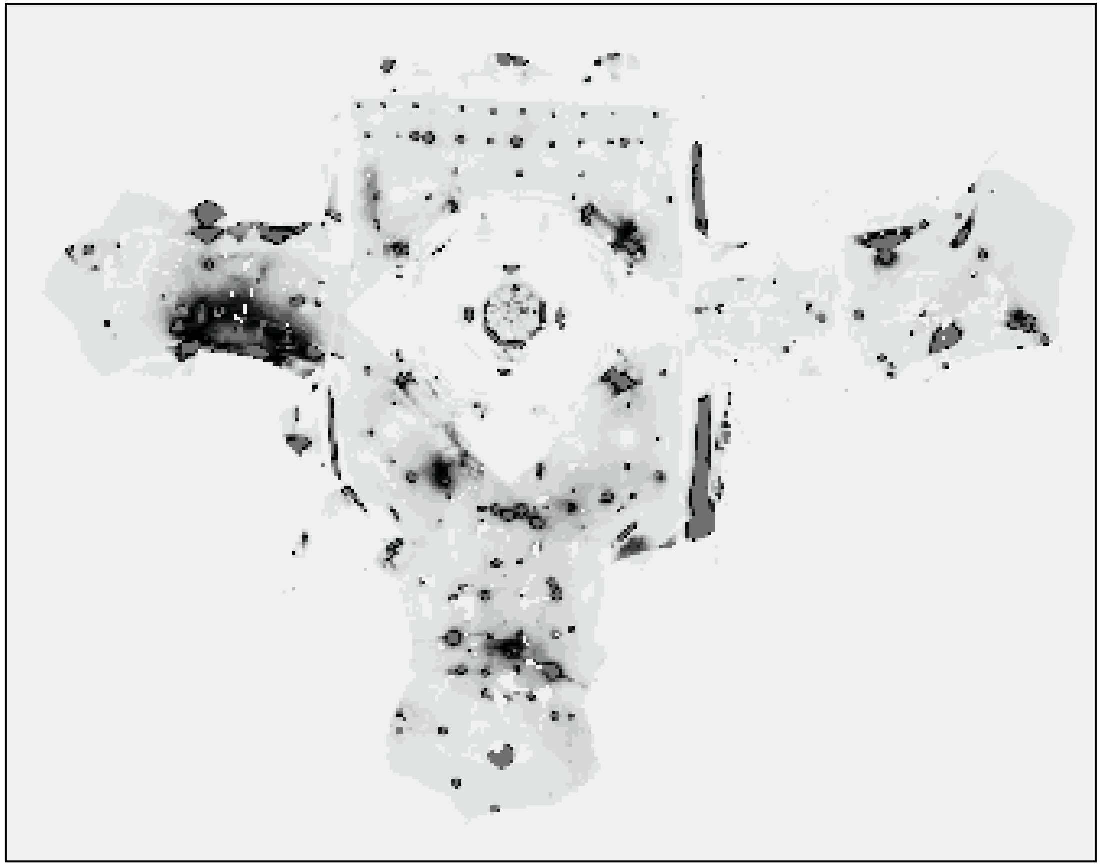


Next to a Light

Measurement z:



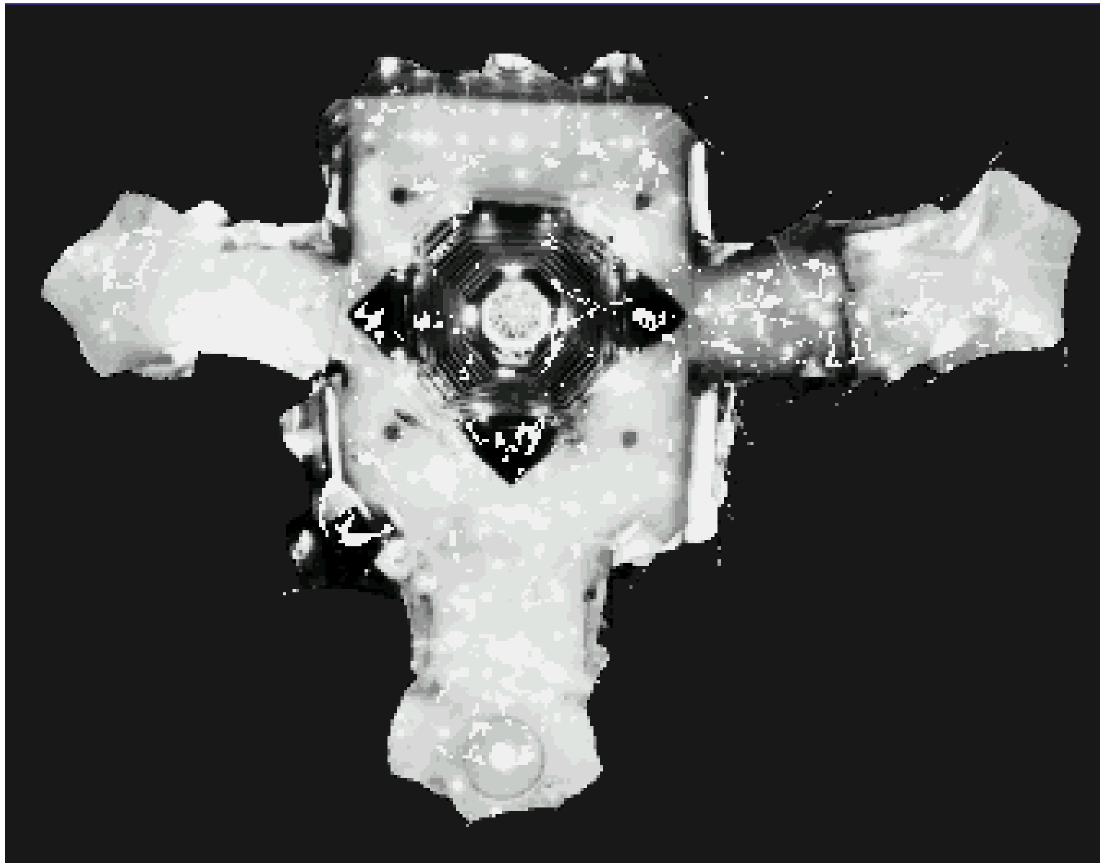
***P(z|x)*:**

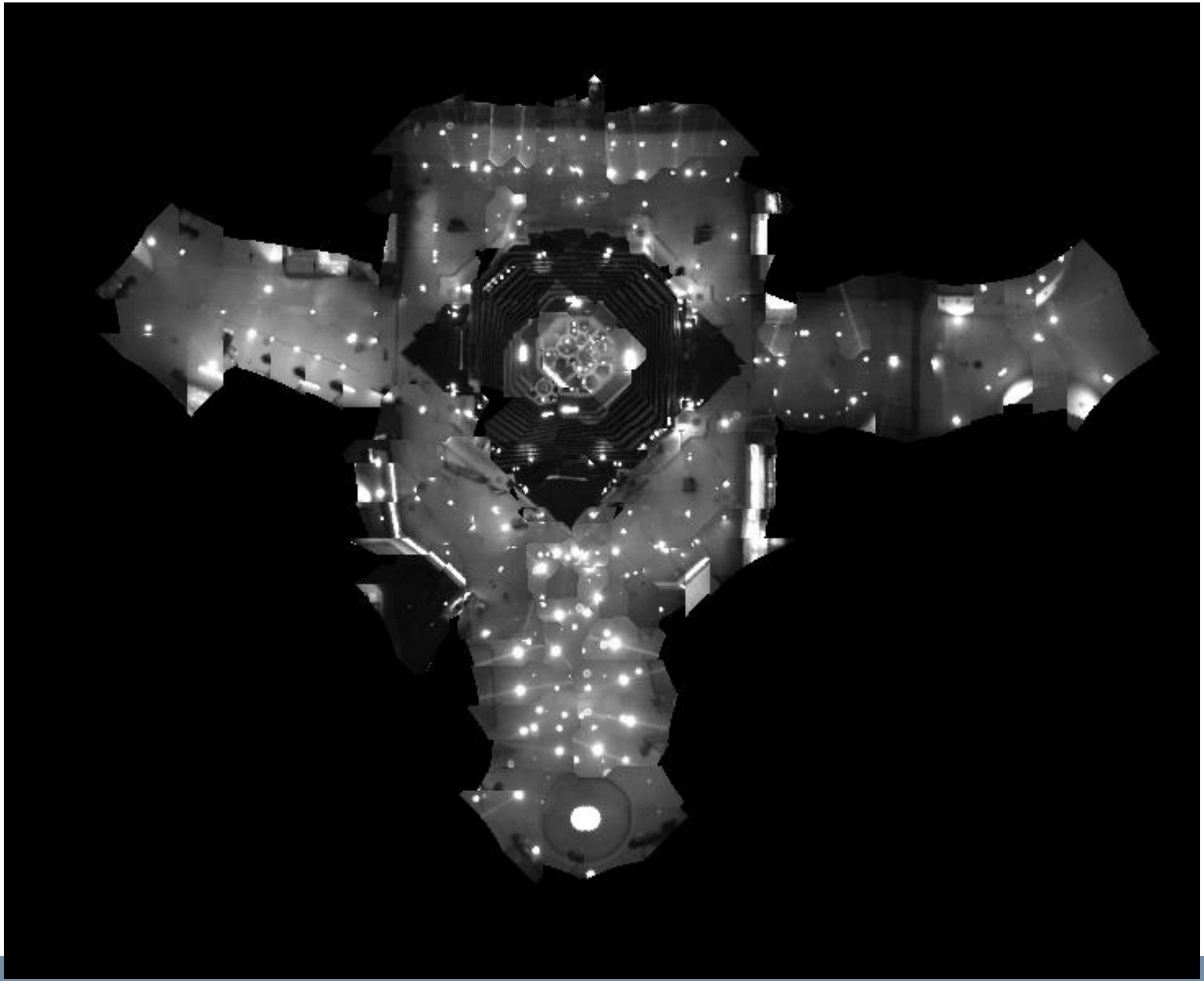


Elsewhere

Measurement z:

$P(z|x)$:



Global Localization Using Vision