Introduction to fuzzy sets

Andrea Bonarini

Artificial Intelligence and Robotics Lab
Department of Electronics and Information
Politecnico di Milano

E-mail: bonarini@elet.polimi.it
URL:http://www.dei.polimi.it/people/bonarini

A bit of history

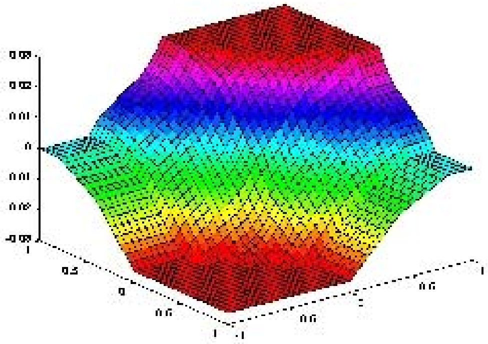
- Fuzzy sets have been defined by Lotfi Zadeh in 1965, as a tool to model approximate concepts
- In 1972 the first “linguistic” fuzzy controller is implemented
- In the Eighties boom of fuzzy controllers first in Japan, then USA and Europe
- In the Nineties applications in many fields: fuzzy data bases, fuzzy decision making, fuzzy clustering, fuzzy learning classifier systems, neuro-fuzzy systems...
  Massive diffusion of fuzzy controllers in end-user goods
- Now, fuzzy systems are the kernel of many “intelligent” devices

Main characteristics

Fuzzy sets**:**
precise model in a finite number of points, smooth transition (approximation) among them.

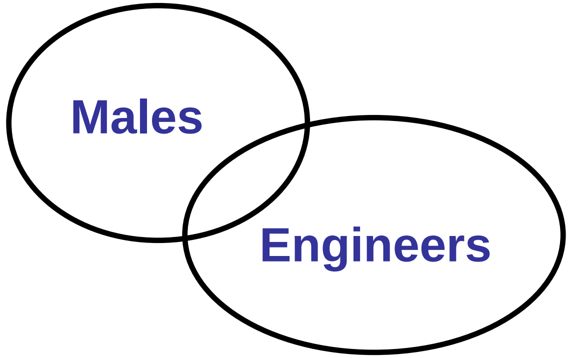
E.g.: control of a power plant**.**

We can define what to do in standard operating conditions (e.g., steam temperature =120°, steam pressure 2 atm), and when in critical situations (e.g., steam temperature= 100°), and design a model that smoothly goes from one point to the other.

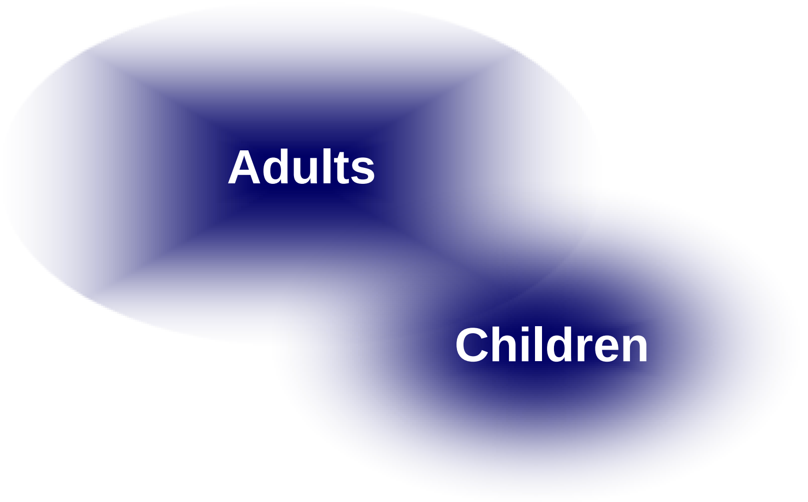


What is a fuzzy set?

A fuzzy set is a set whose membership function may range on the interval [0,1].



Crisp sets



Fuzzy sets

 © A. Bonarini (bonarini@elet.polimi.it)

Fuzzy membership functions

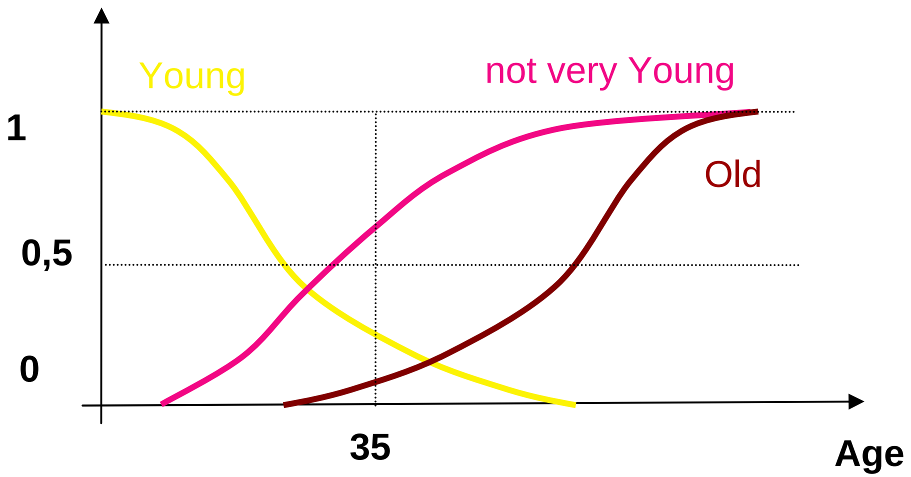
A membership function defines a set

Defines the degree of membership of an element to the set

μ: U → [0, 1]

A 35 years old person is:

- Young **with membership 0,3**
- Old **with membership 0,2**
- not very Young **with membership 0,6**

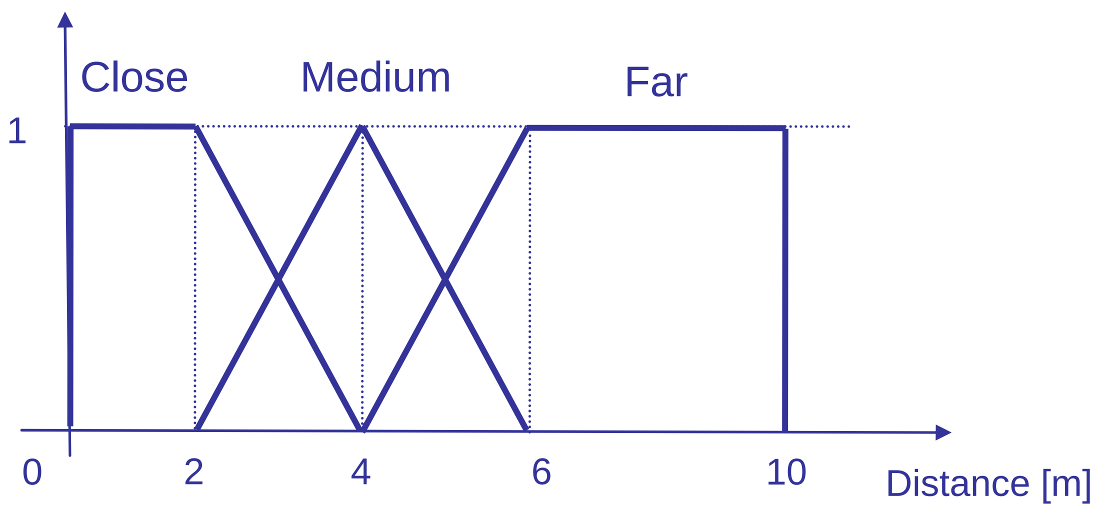


How to define MFs

1. **Select a variable**
2. **Define the range of the variable**
3. **Identify labels**
4. **For each label identify characteristic points**
5. **Identify function shapes**
6. **Check**

Let's try to define some MFs

First of all, the variable… →	**Distance**
Range of the variable →	**[0..10]**
Labels →	**Close, Medium, Far**
Characteristic points →	**0, max, where MF=1, ...**
Function shape →	**Linear**



MFs and concepts

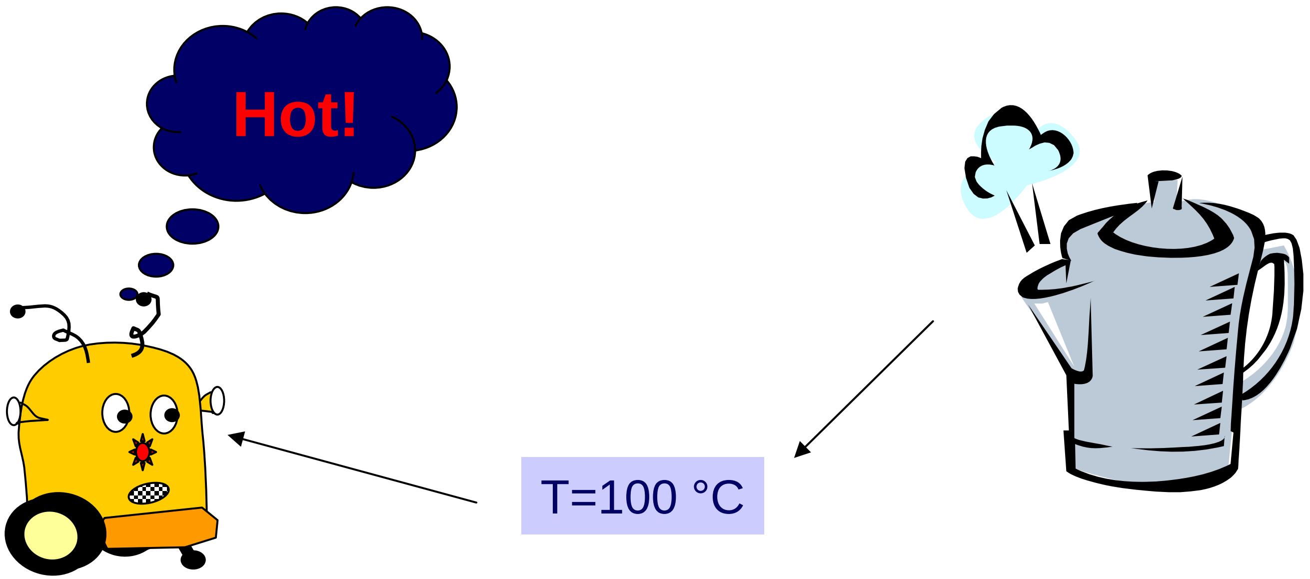
MFs define **fuzzy sets**

Labels denote **fuzzy sets**

Fuzzy sets can be considered as conceptual **representations**

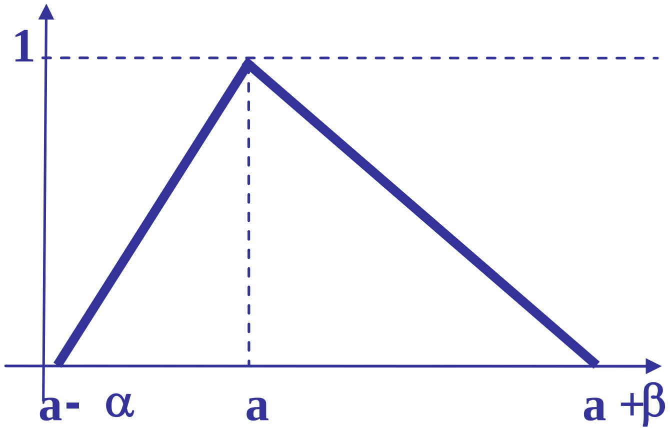
Symbol grounding**:**

> **reason in terms of concepts and ground them on objective reality**

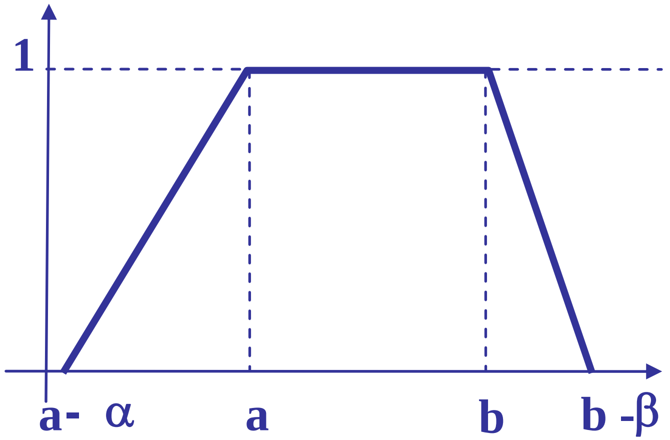


Some conceptual differences

A fuzzy set with only one member with the maximum membership

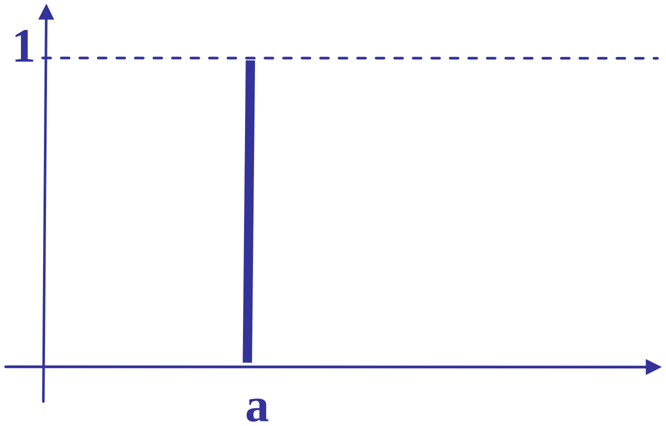


A fuzzy set with a set of members with the maximum membership

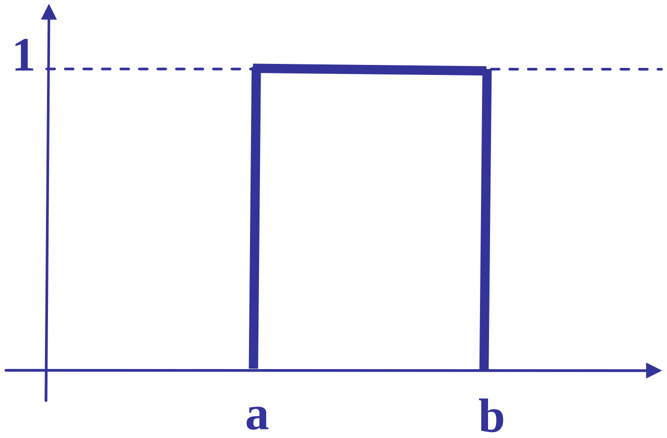


Some conceptual differences

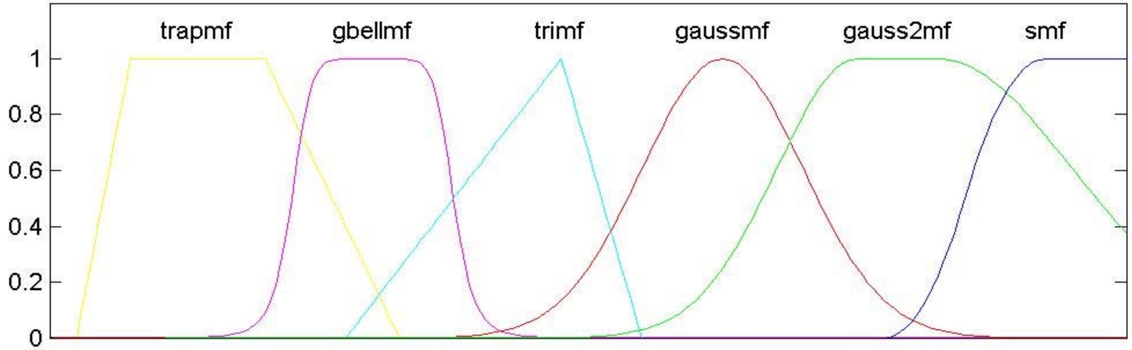
A fuzzy set with only one member

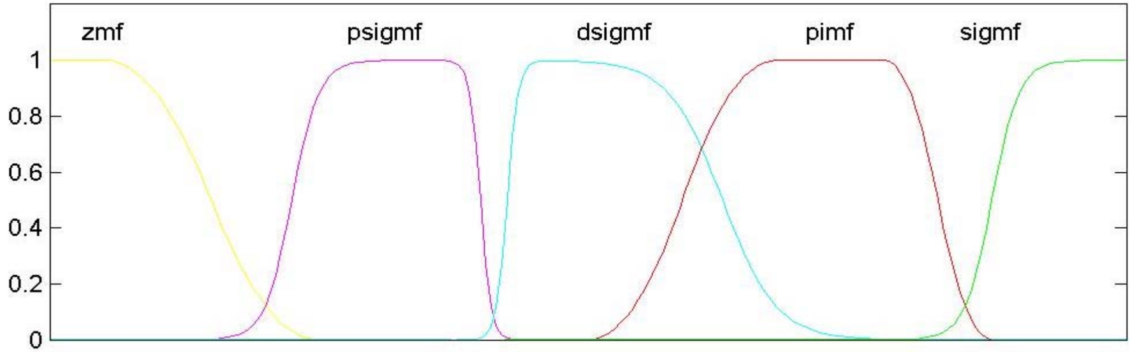


A fuzzy set with all the members having the maximum membership

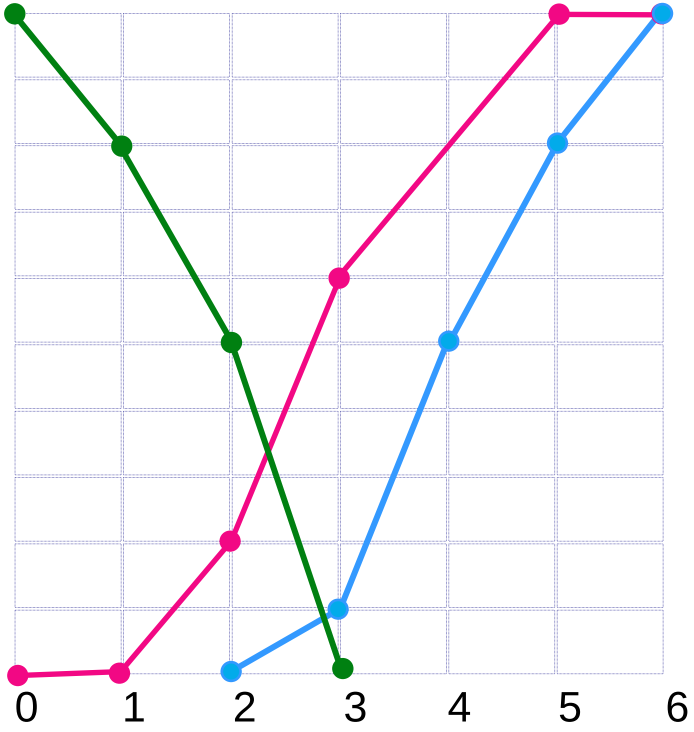


Some variations

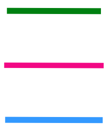




Fuzzy sets on ordinal scales

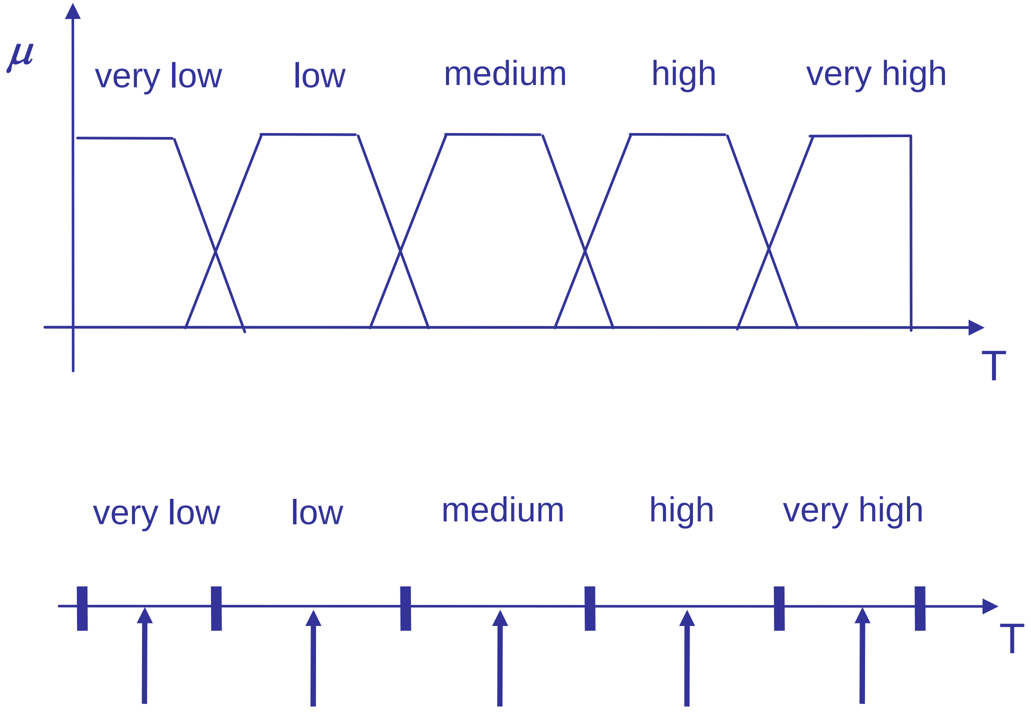


0 - no education
1 - elementary school
2 - high school
3 - two year college
4 - bachelor's degree
5 - masters's degree
6 - doctoral degree



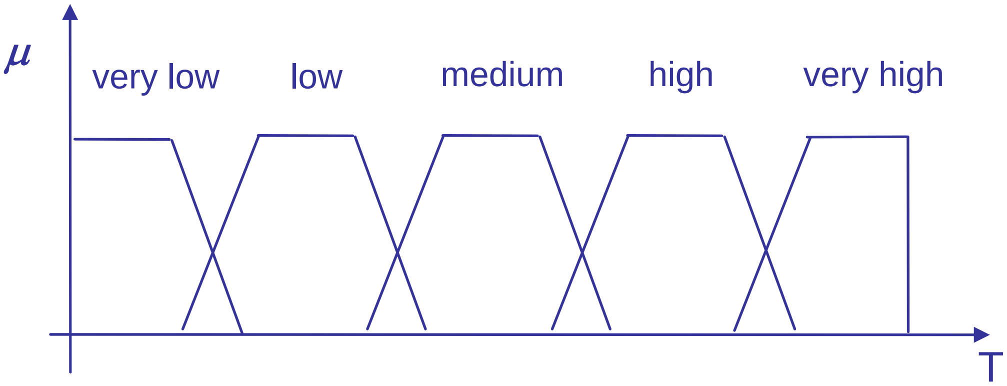
Fuzzy sets and intervals

Smoother transition in labeling a value



Frame of cognition

Fuzzy sets covering the universe of discourse



Each fuzzy set is a *granule*

Properties of a frame of cognition

Coverage

Each element of the universe of discourse is assigned to at least a granule with membership > 0

Unimodality **of fuzzy sets**

There is a unique set of values for each granule with maximum membership

Fuzzy partition**:**

for each value of the universe of discourse the sum of membership degrees to the corresponding granules is 1

Robustness

Let's consider a punctual error as the sum of the errors in interpretation of a point by fuzzy sets due to imprecise measurements, noise, ...

$$e(\hat{a}) = |\mu_1(\hat{a}) - \mu_1(a')| + \ldots + |\mu_n(\hat{a}) - \mu_n(a')|$$

and the integral error, as the integral of e(a) over the range of a

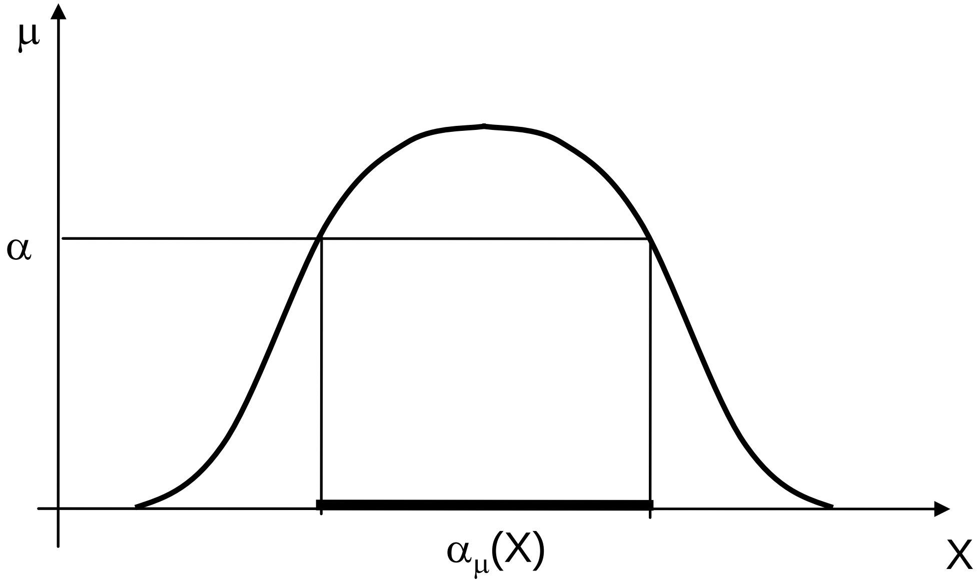
$$e_i = \int e(a)\, da$$

It can be demonstrated that the integral error of a fuzzy partition is smaller than that of a boolean partition, and that it is minimum w.r.t. any other frame of cognition.

α-cuts

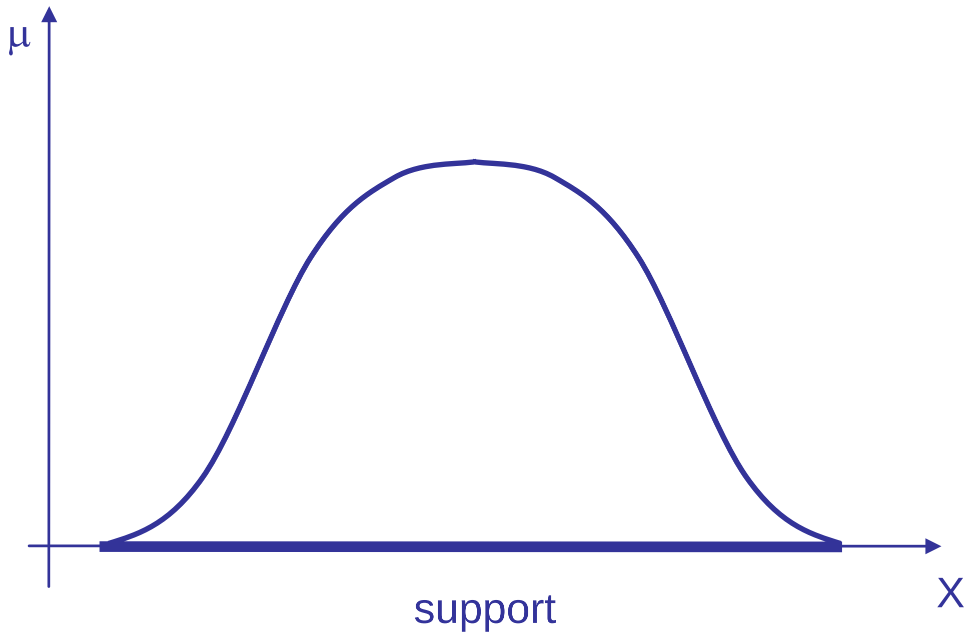
The α-cut of a fuzzy set is the crisp set of the values of x such that $\mu(X) \geq \alpha$

$$\alpha_\mu(X) = \{x \mid \mu(x) \geq \alpha\}$$



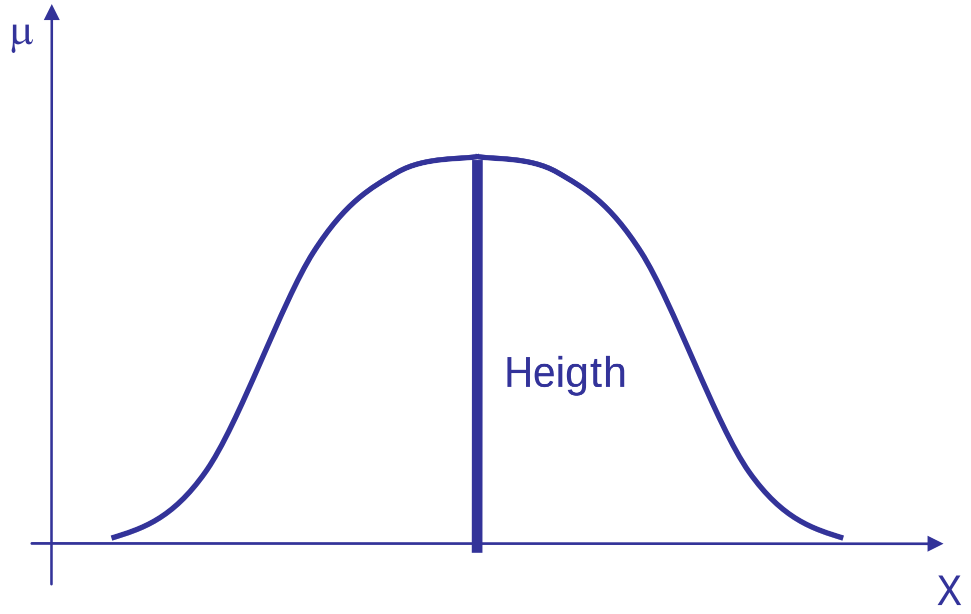
Support of a fuzzy set

The crisp set of values x of X such that $\mu_f(x) > 0$ is the support of the fuzzy set f on the universe X



Height of a fuzzy set

The height *h(A)* of a fuzzy set A on the universe X is the highest membership degree of an element of X to the fuzzy set



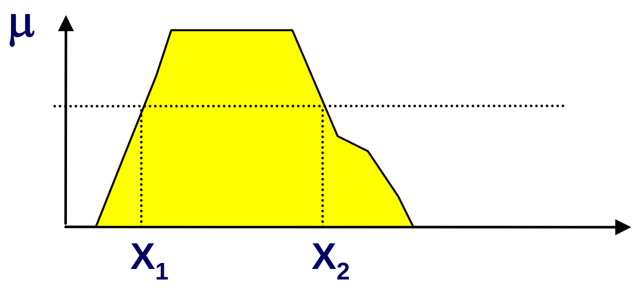
A fuzzy set f is *normal* iff $h_f(x) = 1$

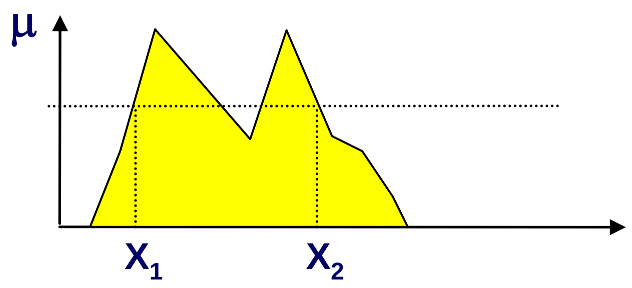
Convex fuzzy sets

A fuzzy set is *convex* iff

$$\mu (\lambda x_1 + (1-\lambda) x_2) \geq \min [\mu (x_1), \mu (x_2)]$$

for any x_1, x_2 in $\Re$ and any λ belonging to [0,1]





Standard operators on fuzzy sets

Complement

$$\mu_{\neg f}(x) = 1 - \mu_f(x)$$

Union

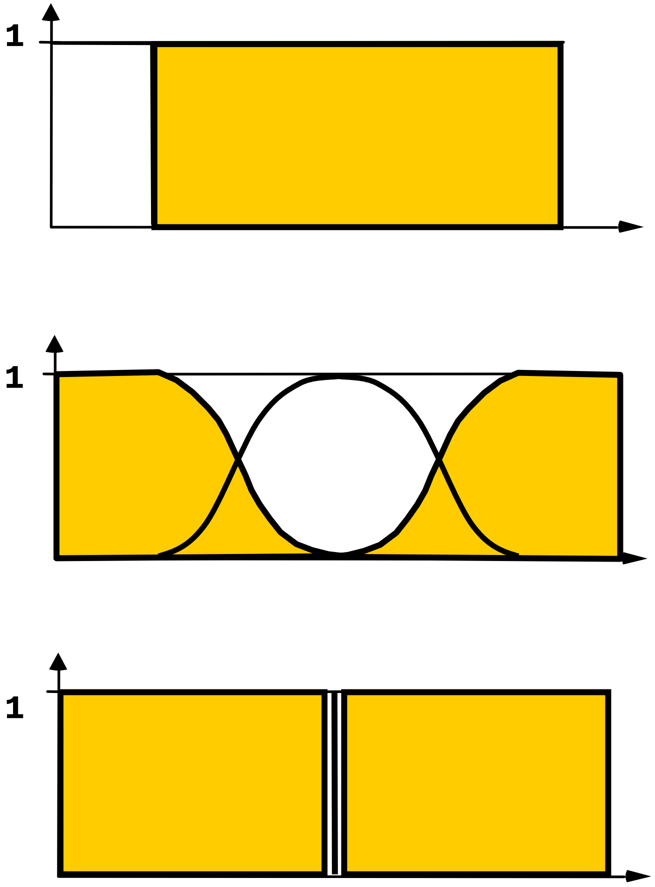
$$\mu_{f_1 \cup f_2}(x) = \max(\mu_{f_1}(x), \mu_{f_2}(x))$$

Intersection

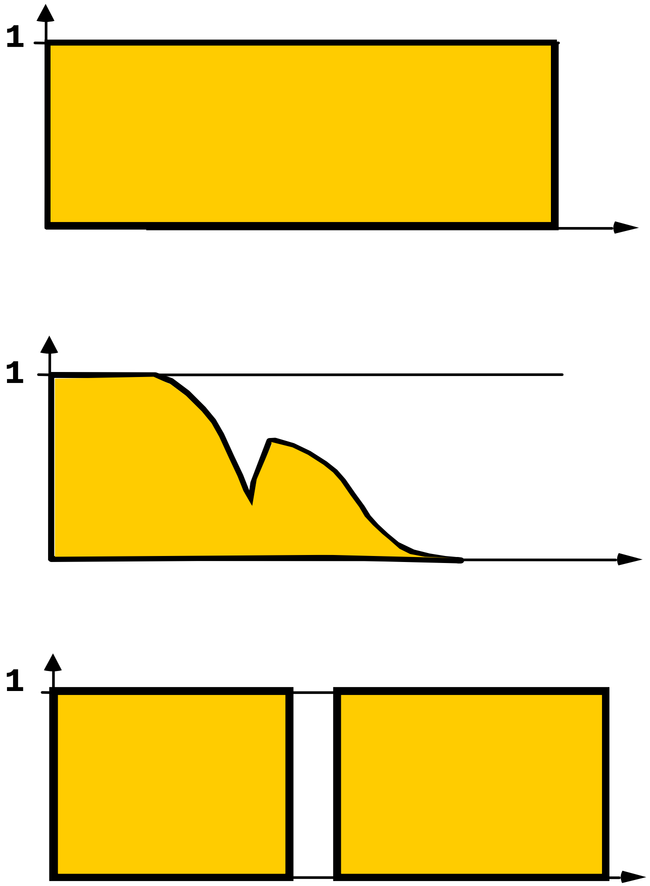
$$\mu_{f_1 \cap f_2}(x) = \min(\mu_{f_1}(x), \mu_{f_2}(x))$$

Examples of operator application

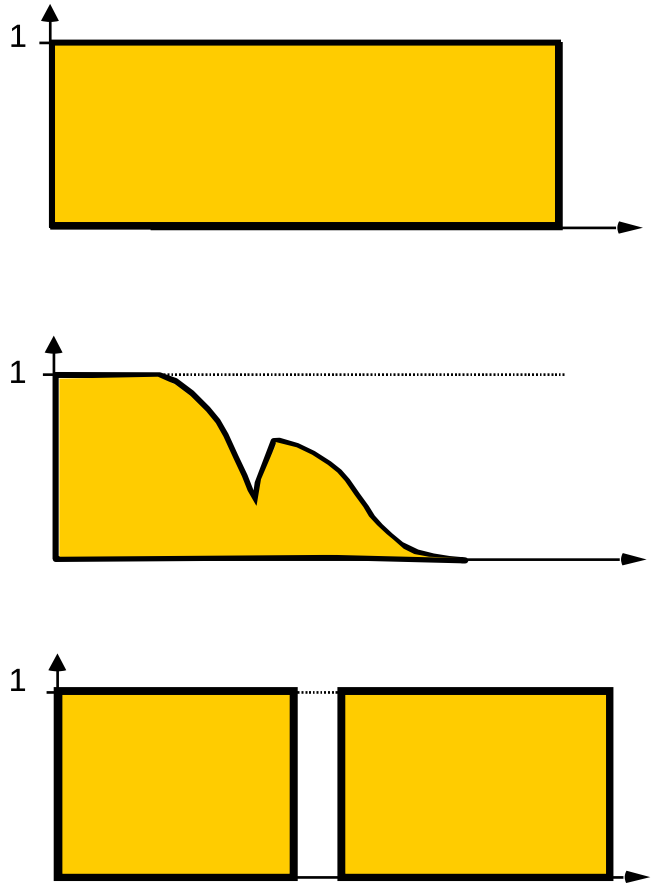
Complement



Union

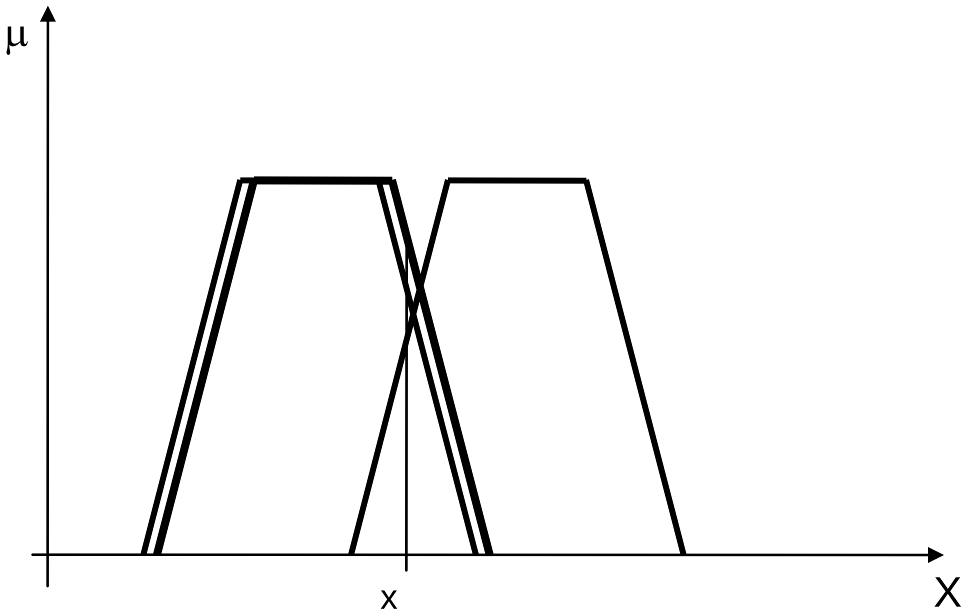


Union



Fundamental property of standard operators

Using the standard operators the maximum error is the one we have on the operand's MFs



Complement

$$c : [0,1] \to [0,1]$$

$$c(\mu_A(x)) = \mu_{\neg A}(x)$$

Axioms:

1. **c(0)=1; c(1)=0 (*boundary conditions*)**
2. **For all a and b in [0,1], if a < b then c(a) ≥ c(b) (*monotonicity*)**
3. **c is a *continuous function***
4. **c is *involutive*, i.e., c(c(a))=a for all a in [0,1]**

Intersection and T-norms

$$\mu_{A \cap B}(x) = i[\mu_A(x), \mu_B(x)]$$

Axioms:

1. **$i[a, 1] = a$ (*boundary conditions*)**
2. **$d \geq b$ implies $i(a,d) \geq i(a,b)$ (*monotonicity*)**
3. **$i(b,a) = i(a,b)$ (*commutativity*)**
4. **$i(i(a,b),d) = i(a,i(b,d))$ (*associativity*)**
5. **i is continuous**
6. **$a \geq i(a,a)$ (*sub-idempotency*)**
7. **$a_1 < a_2$ and $b_1 < b_2$ implies that $i(a_1,b_1) < i(a_2,b_2)$ (*strict monotonicity*)**

T-norms: examples

$$\frac{ab}{\max[\ a, b, \alpha\ ]}$$

for α=1 we have *ab*

for α=0 we have *min(a, b)*

$$t_1(\mu_A(x), \mu_B(x)) = \max(0, \mu_A(x) + \mu_B(x) - 1)$$

$$t_{2.5}(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)}$$

Union and T-conorms (S-norms)

$$\mu_{A\cup B}(x) = u[\mu_A(x), \mu_B(x)]$$

Axioms:

1. **u[a, 0]=a (*boundary conditions*)**
2. **b ≤ d implies u(a,b) ≤ u(a,d) (*monotonicity*)**
3. **u(a,b) = u(b,a) (*commutativity*)**
4. **u(a,u(b,d)) = u(u(a,b),d) (*associativity*)**
5. **u is continuous**
6. **u(a,a) ≥ a (*super-idempotency*)**
7. **$a_1 < a_2$ e $b_1 < b_2$ implies that $u(a_1,b_1)<u(a_2,b_2)$ (strict *monotonicity*)**

T-conorms: examples

$$s(\mu_A(x), \mu_B(x)) = \min\{1, (\mu_A(x)^p + \mu_B(x)^p)^{1/p} \quad p \geq 1$$

$$s_1(\mu_A(x), \mu_B(x)) = \min(1, \mu_A(x) + \mu_B(x))$$

$$s_3(\mu_A(x), \mu_B(x)) = \max(\mu_A(x), \mu_B(x))$$

$$s_+(\mu_A(x), \mu_B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x) * \mu_B(x)$$

Aggregation

$$\mu_A(x) = h[\mu_{A1}(x), \ldots, \mu_{An}(x)]$$

Axioms:

1. **h[0,..., 0]=0, h[1,..., 1]=1 (*boundary conditions*)**
2. ***monotonicity***
3. **h is continuous**
4. **h(a,..,a) = a (*idempotency*)**
5. ***simmetricity***

Properties of aggregation

$$\min(a_1, \ldots, a_n) \le h(a_1, \ldots, a_n) \le \max(a_1, \ldots, a_n)$$

Example of aggregation operator: generalized average

$$h(a_1, \ldots, a_n) = (a_1^{\alpha} + \ldots + a_n^{\alpha})^{1/\alpha} / n$$