POLITECNICO DI MILANO

# Mobile robots kinematics

Matteo Matteucci – matteo.matteucci@polimi.it

# Mobile robots classification



Wheeled robots

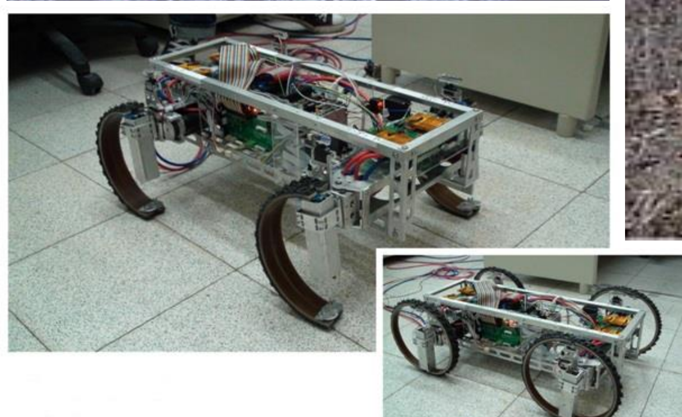
- Kind of wheels
- Kinematics
- Odometry

Legged robots

- Number of legs
- Type of joints
- Stability
- Coordination

Whegs

- ???

# Wheeled Mobile Robots



A robot capable of locomotion on a surface **solely through the actuation of wheel assemblies** mounted on the robot and in contact with the surface. A wheel assembly is a device which provides or allows motion between its mount and surface on which it is intended to have **a single point of rolling contact.**

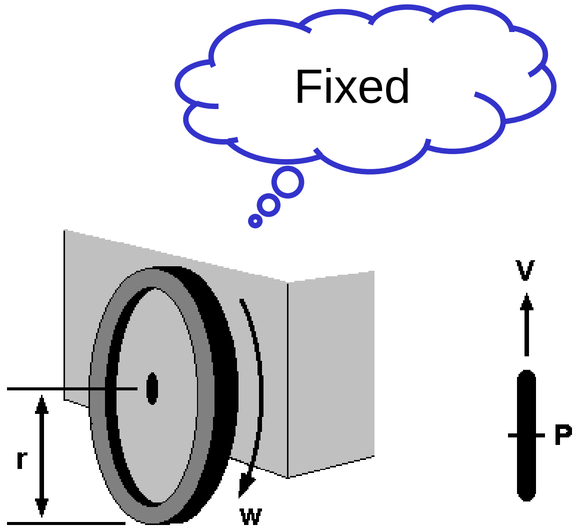
(Muir and Newman, 1986)

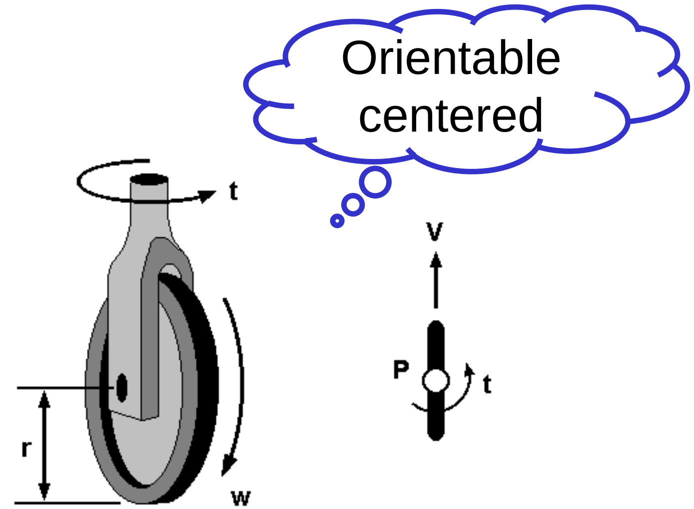
Robot Mobile

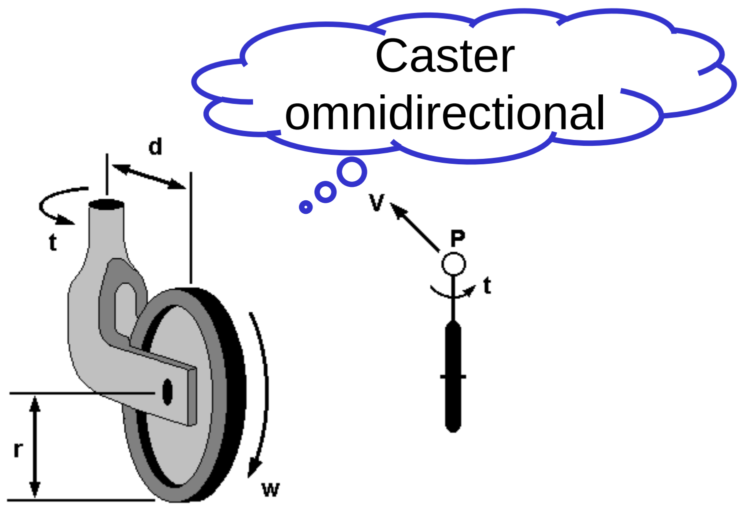
AGV

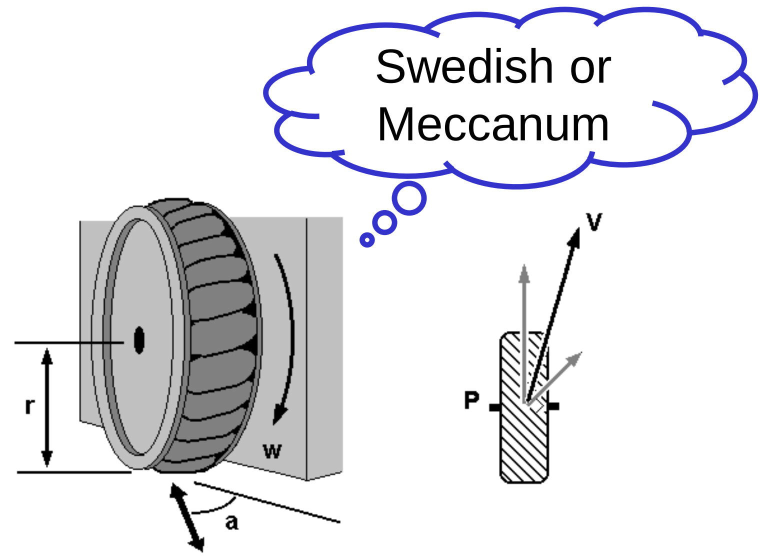
Unmanned vehicle

# Wheels types



# Mobile robots types (some)



Two wheels (differential drive)

- Simple model
- Suffers terrain irregularities
- Cannot translate laterally


Tracks

- Suited for outdoor terrains
- Not accurate movements (with rotations)
- Complex model
- Cannot translate laterally

Omnidirectional (synchro drive)

- Can exploit all degrees of freedom (3DoF)
- Complex model
- Complex structure

# Differential drive

# Differential drive



# Omnidirectional (Swedish wheels)

# Omnidirectional (Syncro drive)

# Degrees of freedom and holonomy



The degrees of freedom are the variables needed to characterize the position of a body in space (a.k.a. *Maneuverability*)

- Differential drive has DOF=3
- Omnidirectional robot has DOF=3

Ability to achieve various poses

The differentiable degrees of freedom (DDoF) are robot independently achievable velocities

- Differential drive has DDoF=2
- Omnidirectional robot has DDoF=3

Ability to achieve various paths

We can have different constraints to the motion

- Holonomic kinematic constraints can be expressed as an explicit function of position variables
- Non-holonomic constraints can be expressed as differential relationship, such as the derivative of a position variable

Constraints can be expressed as a set of equations/disequations of position and velocity of the points in the system

$$\Psi(\dots, P_i, \dot{P}_i, \dots, t) \geq 0$$

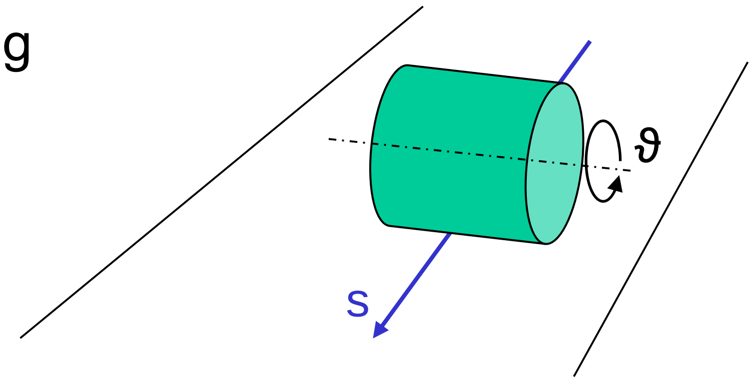
*Holonomic (position)* constraints have no dependence on the velocity

- They subtract a degree of freedom for each constraint equation

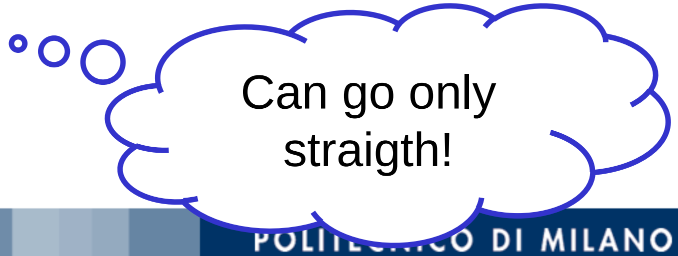
*Non holonomic (mobility)* constraints restrict only the velocity

- They allow to reach any position
- They do not reduce the degrees of freedom
- Some paths are not allowed while any position can be reached (e.g., with a car, whilst it is possible for it to be in any position on the road, it is not possible for it to move sideways)

Let's consider a rolling cylinder without slipping



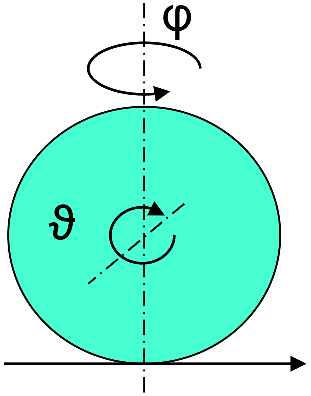

- 6 coordinates, x, y, z, φ, ψ, ϑ
- 5 constraints:
  - z=0, since it rolls on the plane
  - $s = (x^2 + y^2)^{1/2}$, space covered replaces 2 coordinates with 1
  - φ = constant, since we have no slippage
  - ψ = 0,  the plane faces are orthogonal to the plane
  - s' = r ϑ', i.e., ds= r dθ if the cylinder rolls without slipping
- The latter becomes an additional holonomic constrain: $s-s_0= r (\vartheta - \vartheta_0)$
- Only 1 degree of mobility (6 – 5), i.e., s (or ϑ)



# Non holonomic constraint example



Let's consider a thin disk rolling on an horizontal plane




- 6 coordinates, x, y, z, φ, ψ, ϑ
- 4 constraints:
  - z=0, since it rolls on the plane
  - $s = (x^2 + y^2)^{1/2}$, space covered replaces 2 coordinates with 1
  - ψ = 0, the plane faces are orthogonal to the plane
  - s' = r ϑ', i.e., ds= r dθ if the disk rolls without slipping
- It can spin about both ϑ (roll) and, φ (turn) so the latter is non holonomic
- 3 degrees of mobility (6 – 3), i.e., φ + s + ϑ

Can follow any path!

Can go everywhere!

# Some definitions ...

Locomotion: the process of causing an autonomous robot to move

- To produce motion, forces must be applied to the vehicle

Dynamics: the study of motion in which forces are modeled

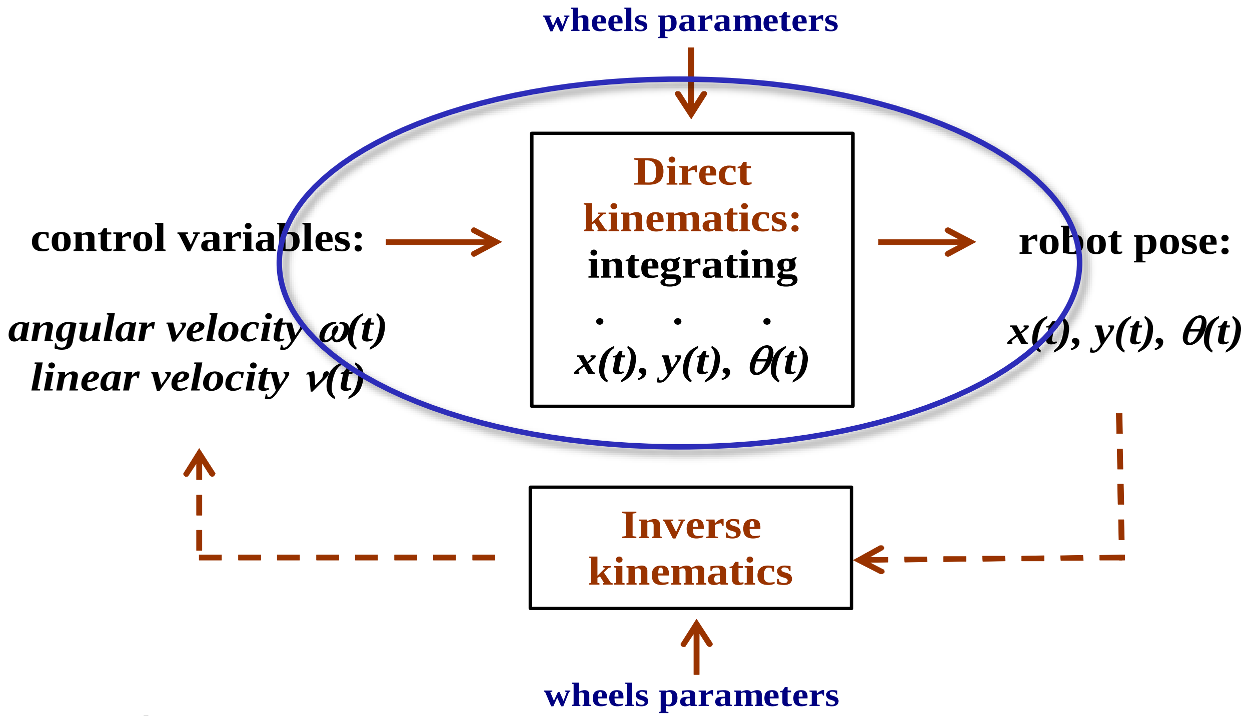
- Includes the energies and speeds associated with these motions

Kinematics: study of motion without considering forces that affect if

- Deals with the geometric relationships that govern the system
- Deals with the relationship between control parameters and the behavior of a system in state space

We'll stick on the plane!

We'll focus on this one!




Direct kinematics

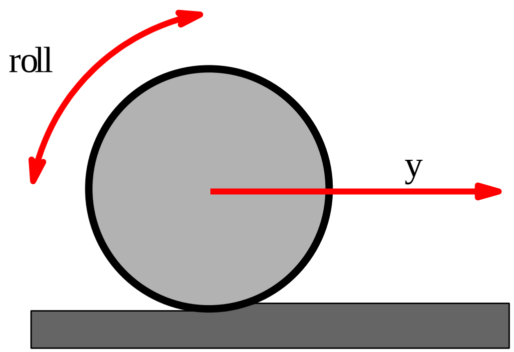
- Given control parameters, e.g., wheels and velocities, and a time of movement *t*, find the pose (x, y, θ) reached by the robot

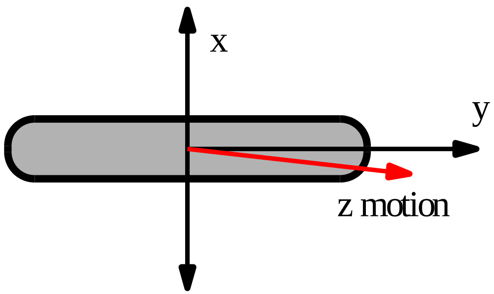
Inverse kinematics

- Given the final pose *(x, y, θ)* find control parameters to move the robot there in a given time *t*

# Wheeled robot assumptions

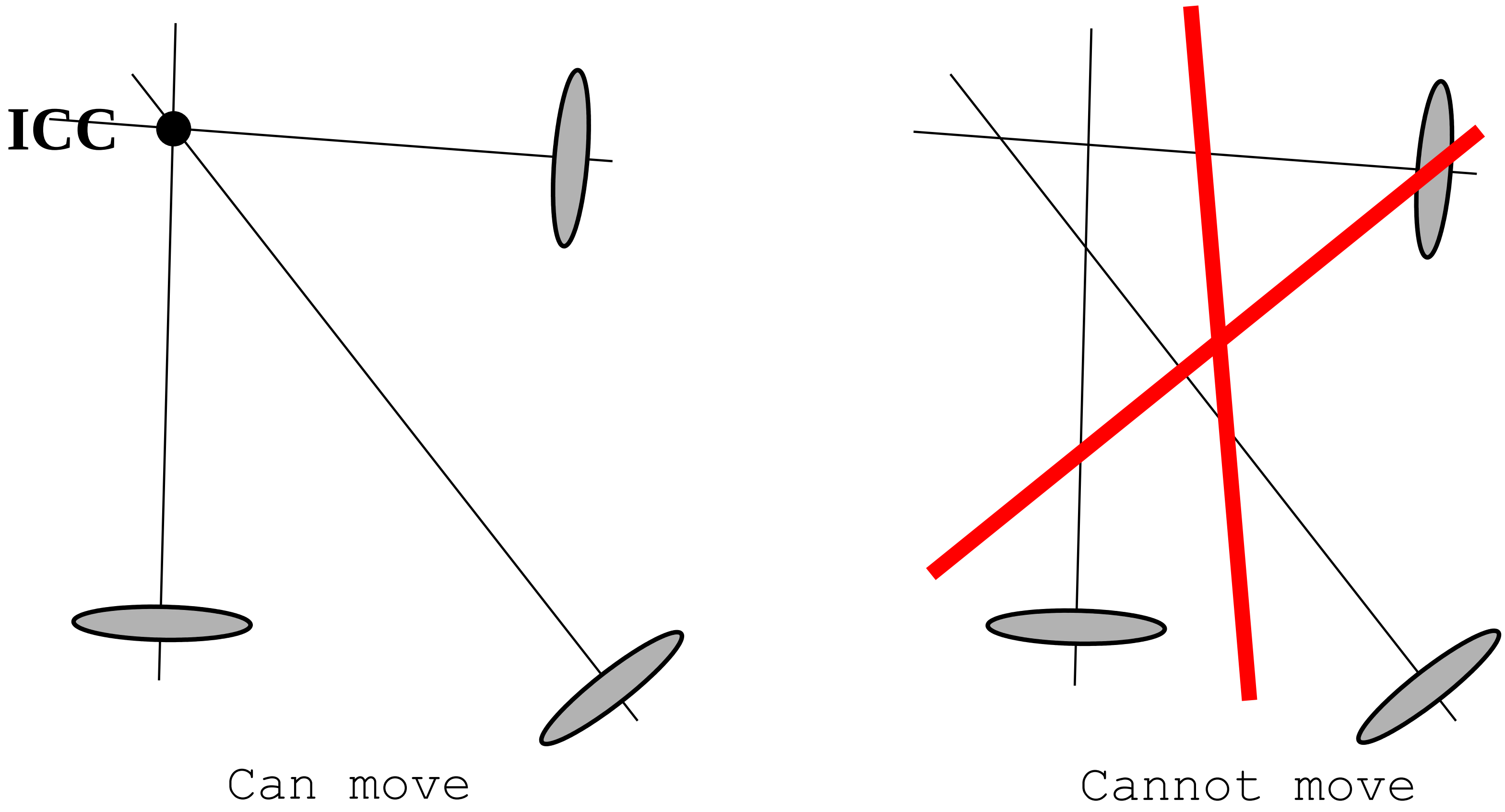
1. Robot made only by rigid parts
2. Each wheel may have a 1 link for steering
3. Steering axes are orthogonal to soil
4. Pure rolling of the wheel about its axis (x axis)
5. No translation of the wheel



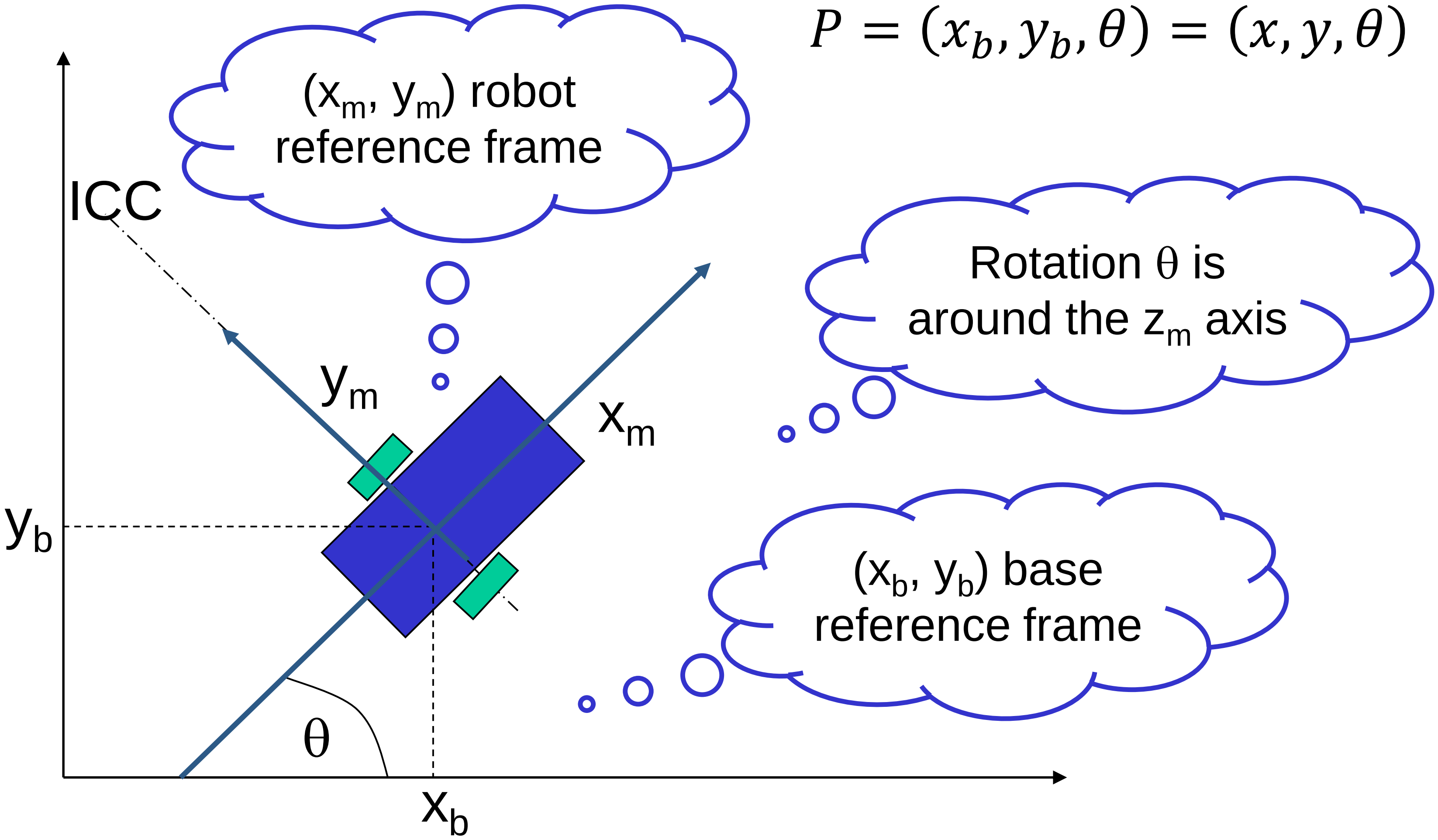

Wheel parameters:
*r* = *radius*
*v* = *linear velocity*
$\omega$ = *angular velocity*

For a robot to move on the plane (3DoF), without slipage, wheels axis have to intersect in a single point named Instantaneous Center of Curvature (ICC) or Instantaneous Center of Rotation (ICR)



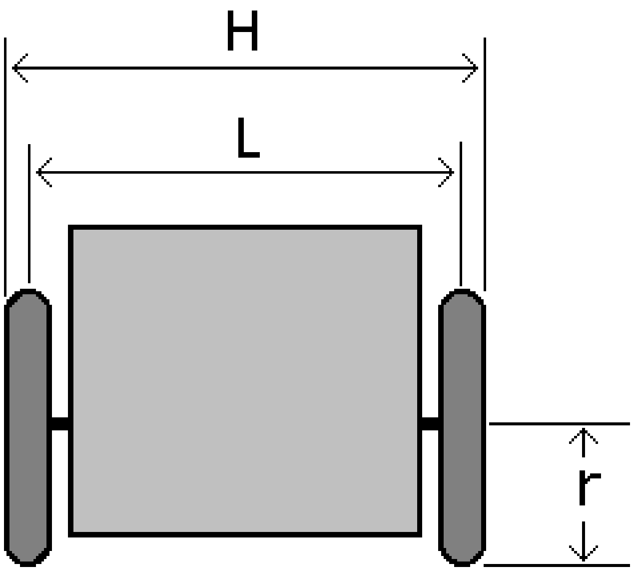
# Representing a pose

$$P = (x_b, y_b, \theta) = (x, y, \theta)$$



Construction

- 2 wheels on the same axis
- 2 independent motors (one for wheel)
- 3rd passive supporting wheel




Variables independently controlled

- $V_R$: velocity of the right wheel
- $V_L$: velocity of the left wheel

Pose representation in base reference: $P = (x_b, y_b, \theta) = (x, y, \theta)$

Control input are:

- v: linear velocity of the robot
- ω: angular velocity of the robot

Linearly related to $V_R$ and $V_L$ ...

# Differential drive kinematics (2)

Right and left wheels follow a circular path with $\omega$ angular velocity and different curvature radius

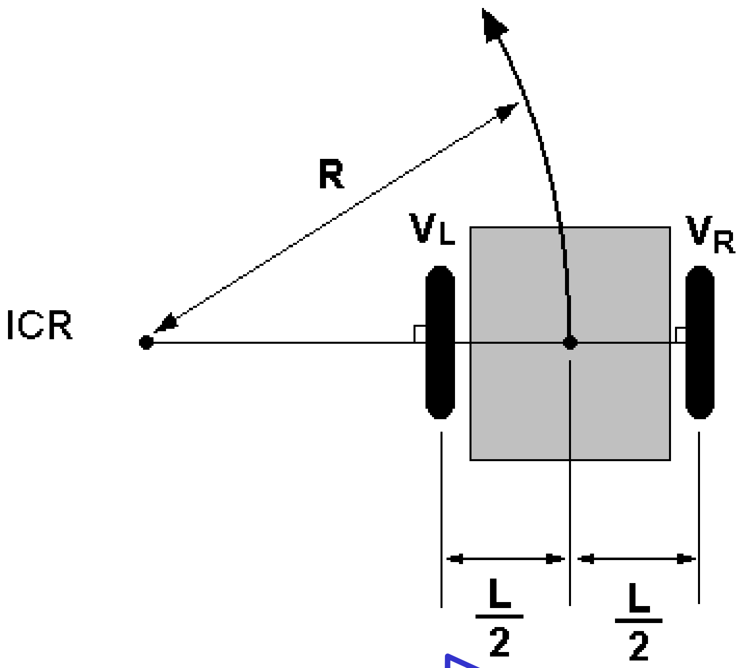
$$\omega\,(R + L/2) = V_R$$
$$\omega\,(R - L/2) = V_L$$

Given $V_R$ and $V_L$ you can find $\omega$ solving for R and equating

$$\omega = V_R - V_L / L$$

Similarly you can find R solving for $\omega$ and equating

$$R = L/2\ (V_R + V_L) / (V_R - V_L)$$




Rotation in place

$R = 0,\ V_R = -V_L$

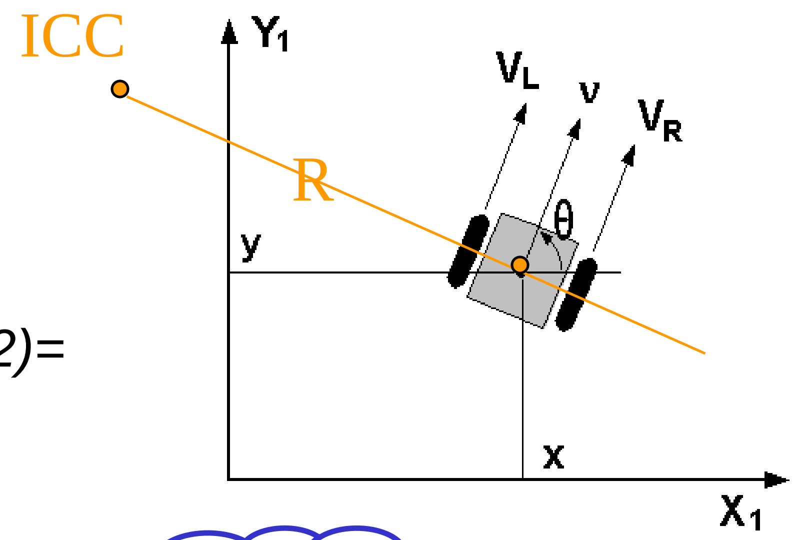
Linear movement

$R = \text{infinite},\ V_R = V_L$

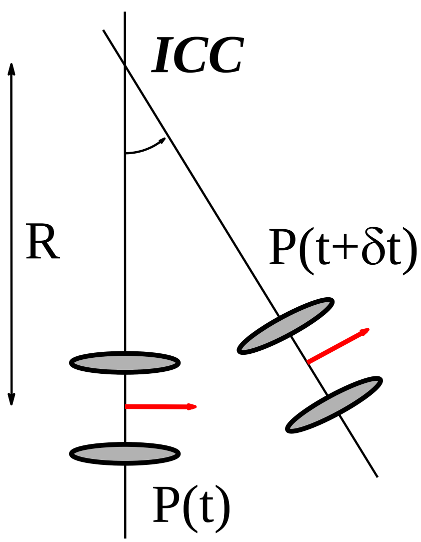
# Differential drive ICC



Wheels move around ICC on a circumference with istantaneous radius R and angular velocity ω




*ICC =* $(x+R\cos(\theta+\pi/2), y+R\sin(\theta+\pi/2)) = (x-R\sin(\theta), y+R\cos(\theta))$

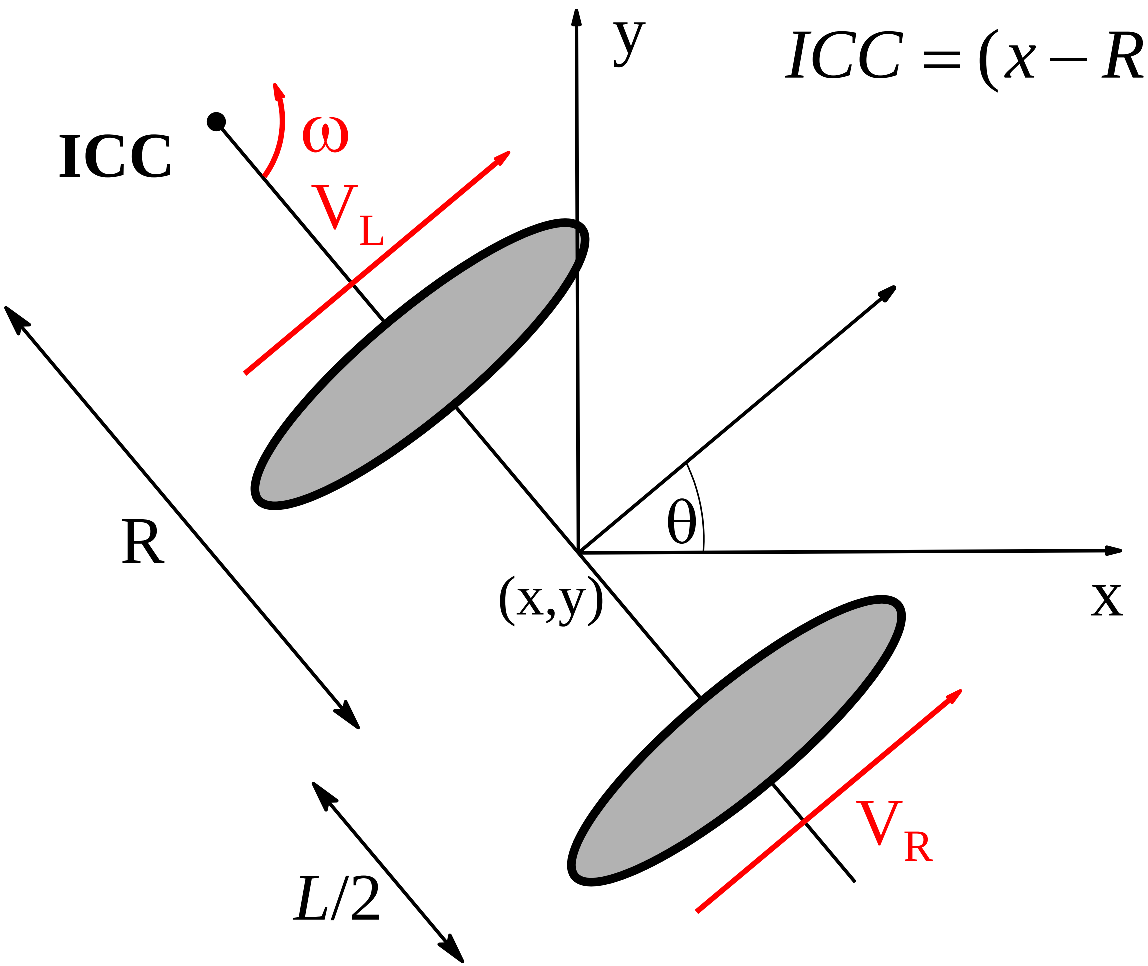



Rotate around ICC

Translate robot in ICC

Translate robot back

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} \cos(\omega\cdot\delta t) & -\sin(\omega\cdot\delta t) & 0 \\ \sin(\omega\cdot\delta t) & \cos(\omega\cdot\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x-ICC_x \\ y-ICC_y \\ \theta \end{bmatrix} + \begin{bmatrix} \mathrm{ICC}_x \\ \mathrm{ICC}_y \\ \omega\cdot\delta t \end{bmatrix}$$




$$ICC = (x - R \cdot \sin(\theta), y + R \cdot \cos(\theta))$$

$$V_R = \omega \cdot (R + L/2)$$

$$V_L = \omega \cdot (R - L/2)$$

$$R = \frac{L}{2} \frac{(V_R + V_L)}{(V_R - V_L)}$$

$$V = \frac{V_R + V_L}{2}$$

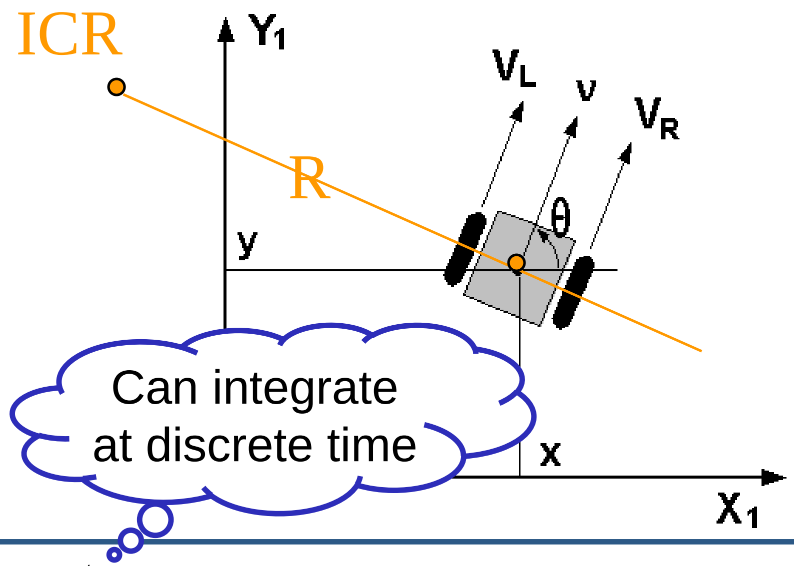
$$\omega = \frac{V_R - V_L}{L}$$

Being know

$$\omega = (V_R - V_L) / L$$

$$R = L/2 (V_R + V_L) / (V_R - V_L)$$

$$V = \omega R = (V_R + V_L) / 2$$




Compute the velocity in the base frame

$$V_x = V(t) \cos(\theta(t))$$

$$V_y = V(t) \sin(\theta(t))$$

Integrate position in base frame

$$x(t) = \int V(t) \cos(\theta(t))\, dt$$

$$y(t) = \int V(t) \sin(\theta(t))\, dt$$

$$\theta(t) = \int \omega(t)\, dt$$

$$x(t) = \frac{1}{2}\int_0^t (V_R(t') + V_L(t')) \cdot \cos(\theta(t')) \cdot dt'$$

$$y(t) = \frac{1}{2}\int_0^t (V_R(t') + V_L(t')) \cdot \sin(\theta(t')) \cdot dt'$$

$$\theta(t) = \frac{1}{L}\int_0^t (V_R(t') - V_L(t')) \cdot dt'$$

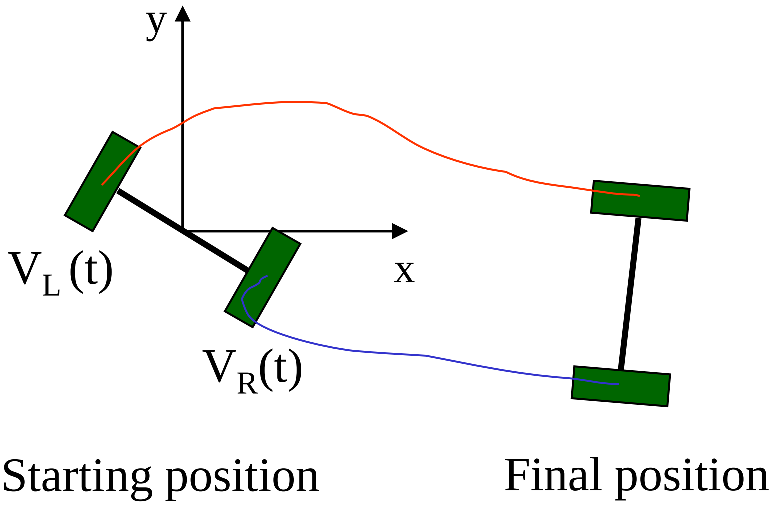
# Inverse kinematics



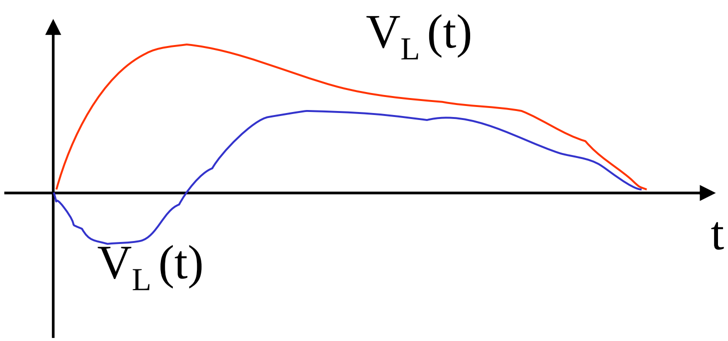
Given a desired position or velocity, what can we do to achieve it?

Finding “some” solution is not hard, but finding the “best” solution can be very difficult:

- Shortest time
- Most energy efficient
- Smoothest velocity profiles



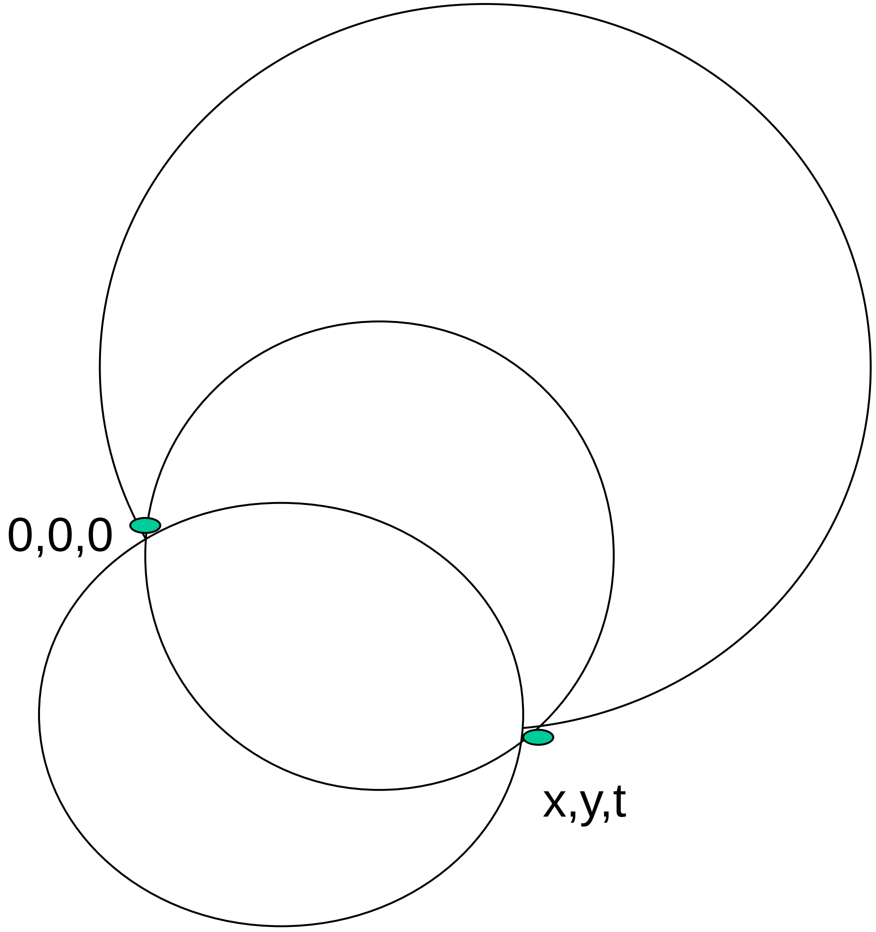

Moreover we have non holonomic constrains and only two control variables; we cannot directly reach any of the 3DoF final positions ...



# Non holonomic constraint

The equations of the direct kinematics describe a constraint on the velocity of the robot that cannot be integrated into a positional constraint (non holonomic constraint):

- The robot moves on a circle passing for (0,0) at time 0 and (x,y) at time t
- Infinite admissible solutions exists, but we want a specific $\theta$
- No independent control of $\theta$ is possible




Nevertheless a straightforward solution exists if we limit the class of control functions for $V_R$ and $V_L$ …

# Differential drive inverse kinematics

Decompose the problem and control only few DoF at the time

1. Turn so that the wheels are parallel to the line between the original and final position of robot origin

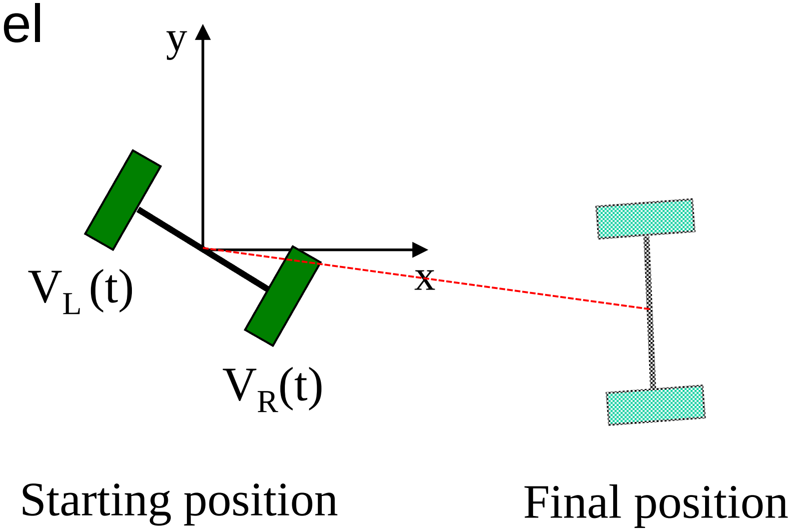
$$-V_L(t) = V_R(t) = V_{max}$$

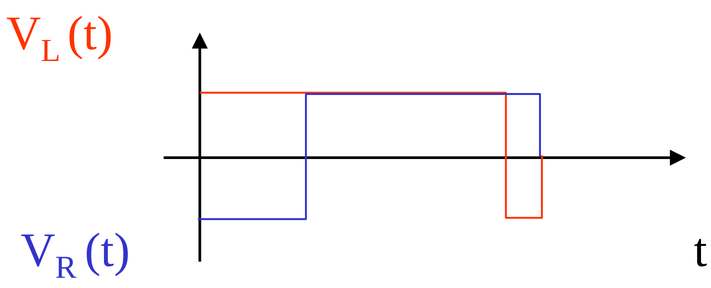
2. Drive straight until the robot's origin coincides with destination

$$V_L(t) = V_R(t) = V_{max}$$

3. Rotate again in to achieve the desired final orientation

$$-V_L(t) = V_R(t) = V_{max}$$

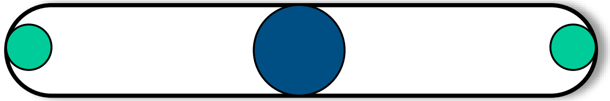


# Vehicles with tracks

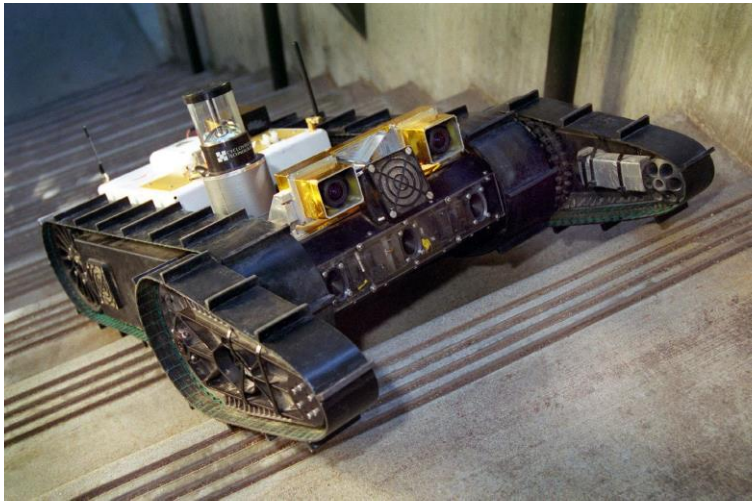
Vehicles with track have a kinematics similar to the differential drive

- Speed control of each track
- Use the height of the track as wheel diameter



- Often named Skid Steering

Need proper calibration and slippage modeling

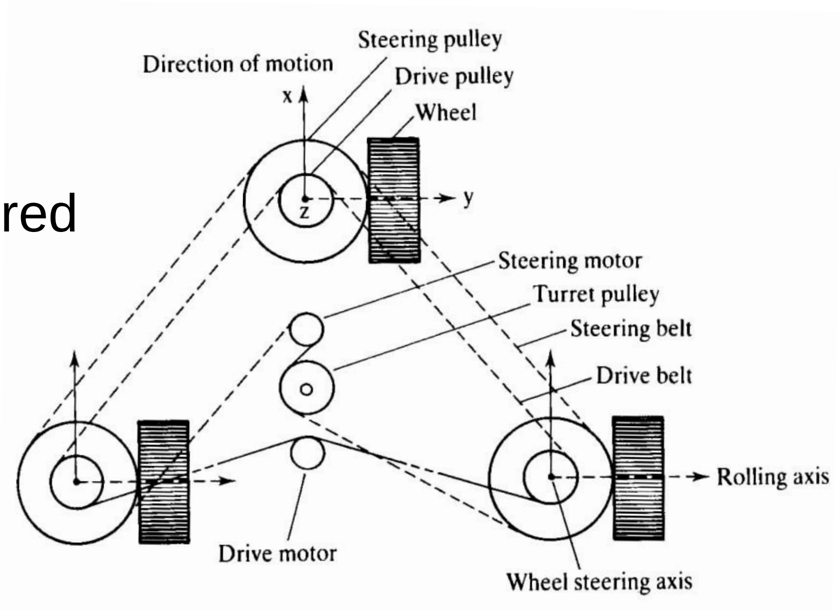




# Synchronous drive

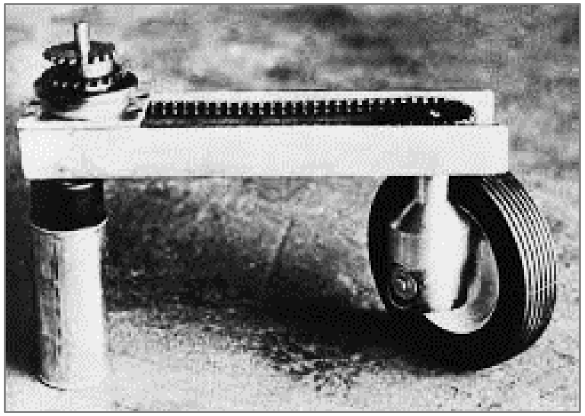
Complex mechanical robot design

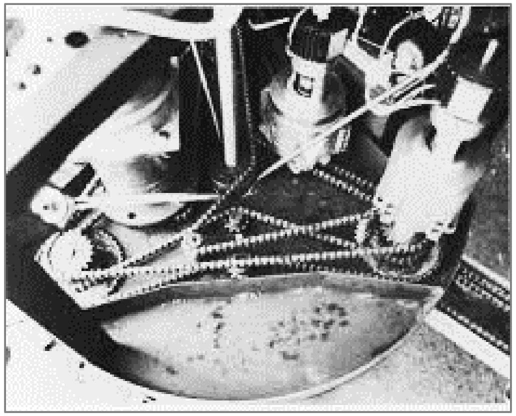
- (At least) 3 wheels actuated and steered
- A motor to roll all the wheels, 
  a second motor to rotate them
- Wheels point in the same direction
- It is possible to control directly $\theta$



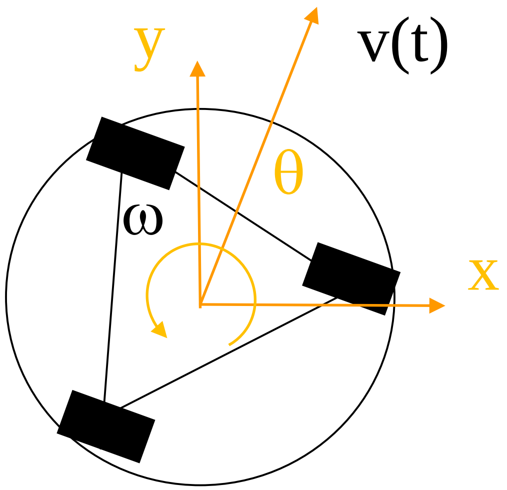

Robot control variables

- Linear velocity v(t)
- Angular velocity $\omega$(t)





Its ICC is always at the infite and the robot is **holonomic**




Robot control for the synchronous drive

- Direct control of v(t) and $\omega$(t)
- Steering changes the direction of ICC

Particular cases:

- v(t)=0, $\omega$(t) = $\omega$ for dt → robot rotates in place
- v(t)=v, $\omega$(t) = 0 for dt → robot moves linearly

Compute the velocity in the base frame

$$V_x = V(t) \cos(\theta(t))$$

$$V_y = V(t) \sin(\theta(t))$$

Calles odometry also for diff drive!

Integrate position in base frame to get the robot odometry (traversed path) ...

$$x(t) = \int_0^t v(t') \cos[\theta(t')] dt'$$

$$y(t) = \int_0^t v(t') \sin[\theta(t')] dt'$$

$$\theta(t) = \int_0^t \omega(t') \, dt'$$

Decompose the problem and control only a few degrees of freedom at a time

1. Turn so that the wheels are parallel to the line between the original and final position of robot origin

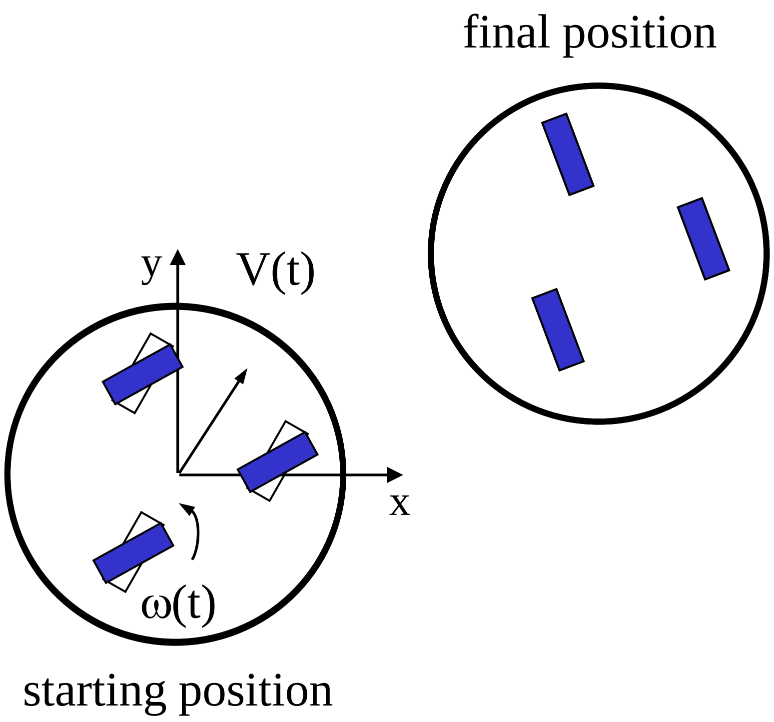
$$\omega(t) = \omega_{max}$$

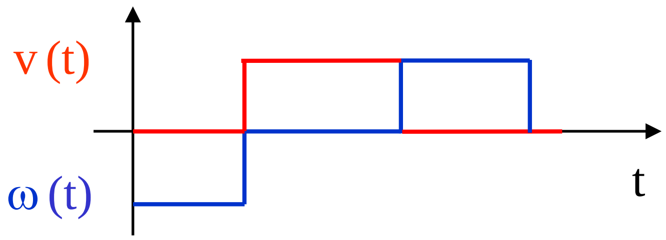
2. Drive straight until the robot's origin coincides with destination

$$v(t) = v_{max}$$

3. Rotate again in to achieve the desired final orientation

$$\omega(t) = \omega_{max}$$



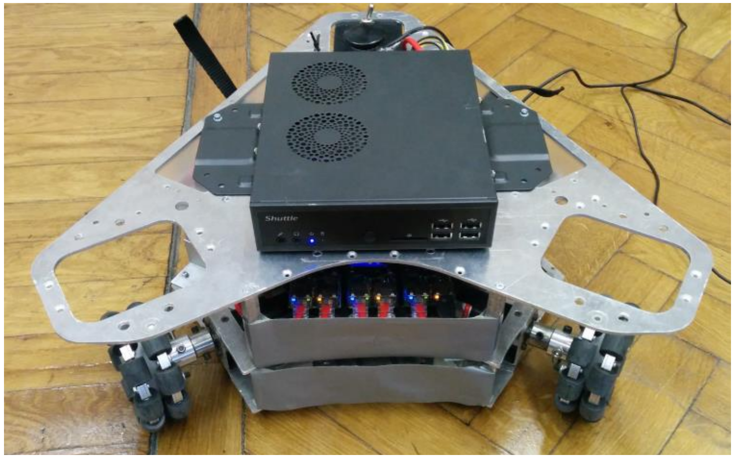



# Omnidirectional (Syncro drive)

# Omnidirectional robot

Simple mechanical robot design

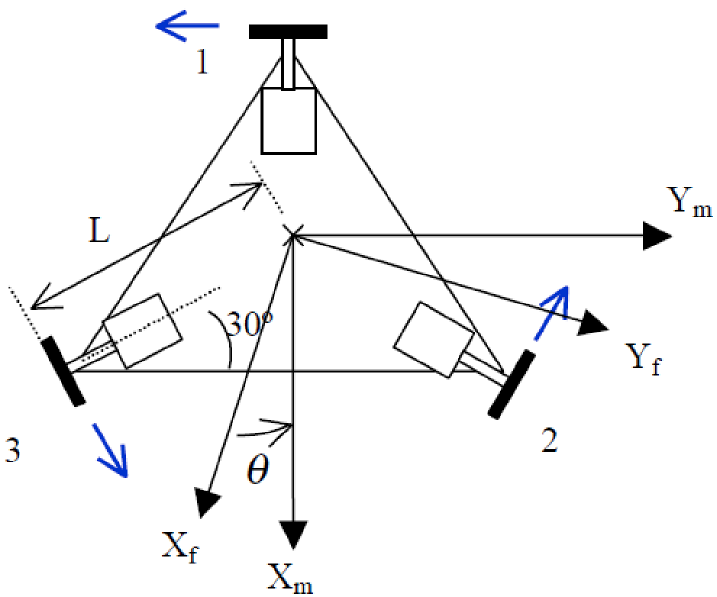
- (At least) 3 Swedish wheels actuated
- One independent motor per wheel
- Wheels point in different direction
- It is possible to control directly x, y, θ



Robot control variables

- Linear velocity v(t) (each component)
- Angular velocity ω(t)

$$\begin{bmatrix} V_x \\ V_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}}r & \frac{1}{\sqrt{3}}r \\ -\frac{2}{3}r & \frac{1}{3}r & \frac{1}{3}r \\ \frac{r}{3L} & \frac{r}{3L} & \frac{r}{3L} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$



# Omnidirectional (Swedish wheels)

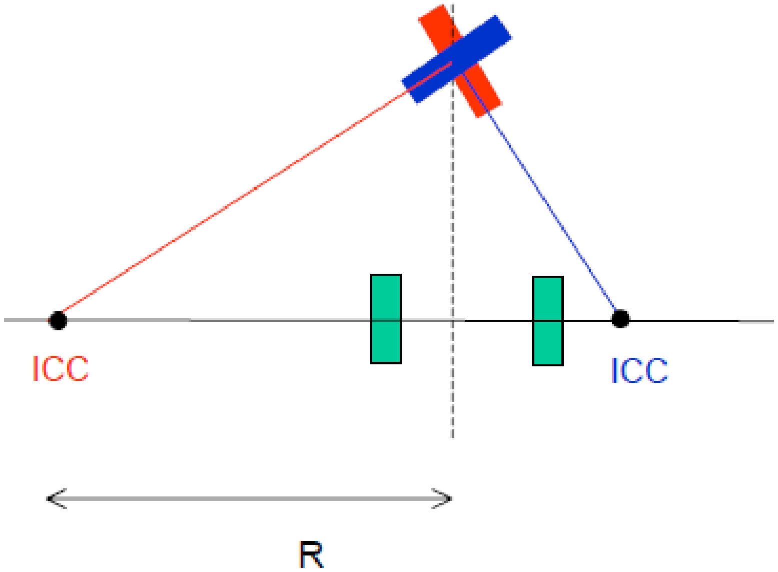
# Tricycle kinematics

The Tricycle is the typical kinematics of AGV

- One actuated and steerable wheel
- 2 additional passive wheels
- Cannot control $\theta$ independently
- ICC must lie on the line that passes through the fixed wheels


Robot control variables

- Steering direction $\alpha(t)$
- Angular velocity of steering wheel $\omega(t)$




Particular cases:

- $\alpha(t)=0$, $\omega(t) = \omega \rightarrow$ moves straight
- $\alpha(t)=90$, $\omega(t) = \omega \rightarrow$ rotates in place

# Tricycle kinematics

Direct kinematics can be derived as:

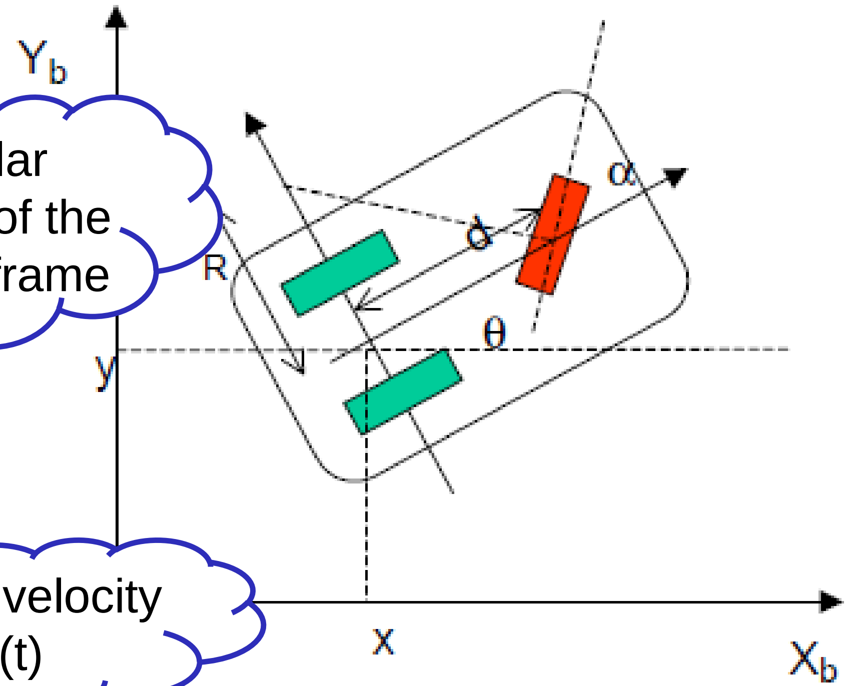
$$r = steering\ wheel\ radius$$

$$V_s(t) = \omega_s(t) \cdot r$$

$$R(t) = d \cdot \tan(\frac{\pi}{2} - \alpha(t))$$

$$\omega(t) = \frac{\omega_s(t) \cdot r}{\sqrt{d^2 + R(t)^2}} = \frac{V_s(t)}{d} \sin \alpha(t)$$

Angular velocity of the moving frame




Linear velocity v(t)

In the robot frame

$$V_x(t) = V_s(t) \cdot \cos \alpha(t)$$

We assume no slipage

$$V_y(t) = 0$$

Angular velocity ω(t)

$$\dot{\theta} = \frac{V_s(t)}{d} \cdot \sin\ \alpha(t)$$

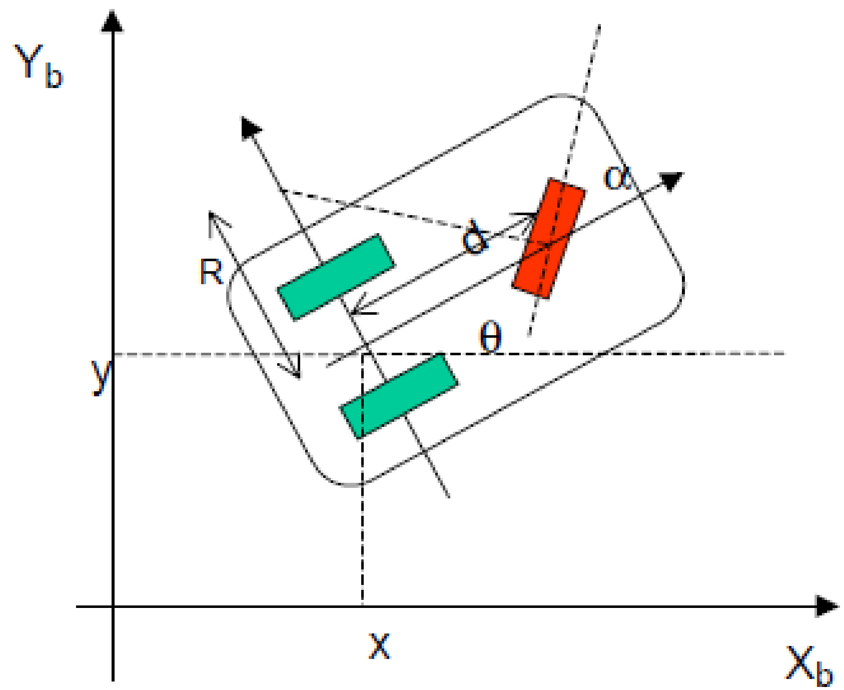
Direct kinematics can be derived as:

$$r = steering\ wheel\ radius$$

$$V_s(t) = \omega_s(t) \cdot r$$

$$R(t) = d \cdot \tan(\frac{\pi}{2} - \alpha(t))$$

$$\omega(t) = \frac{\omega_s(t) \cdot r}{\sqrt{d^2 + R(t)^2}} = \frac{V_s(t)}{d} \sin\alpha(t)$$




In the world frame

$$\dot{x}\,(t) = V_s(t) \cdot \cos\alpha(t) \cdot \cos\theta(t) = V(t) \cdot \cos\theta(t)$$

$$\dot{y}\,(t) = V_s(t) \cdot \cos\alpha(t) \cdot \sin\theta(t) = V(t) \cdot sin\,\theta(t)$$

$$\dot{\theta} = \frac{V_s(t)}{d} \cdot \sin\ \alpha(t) = \omega(t)$$

Most diffused kinematics on the planet

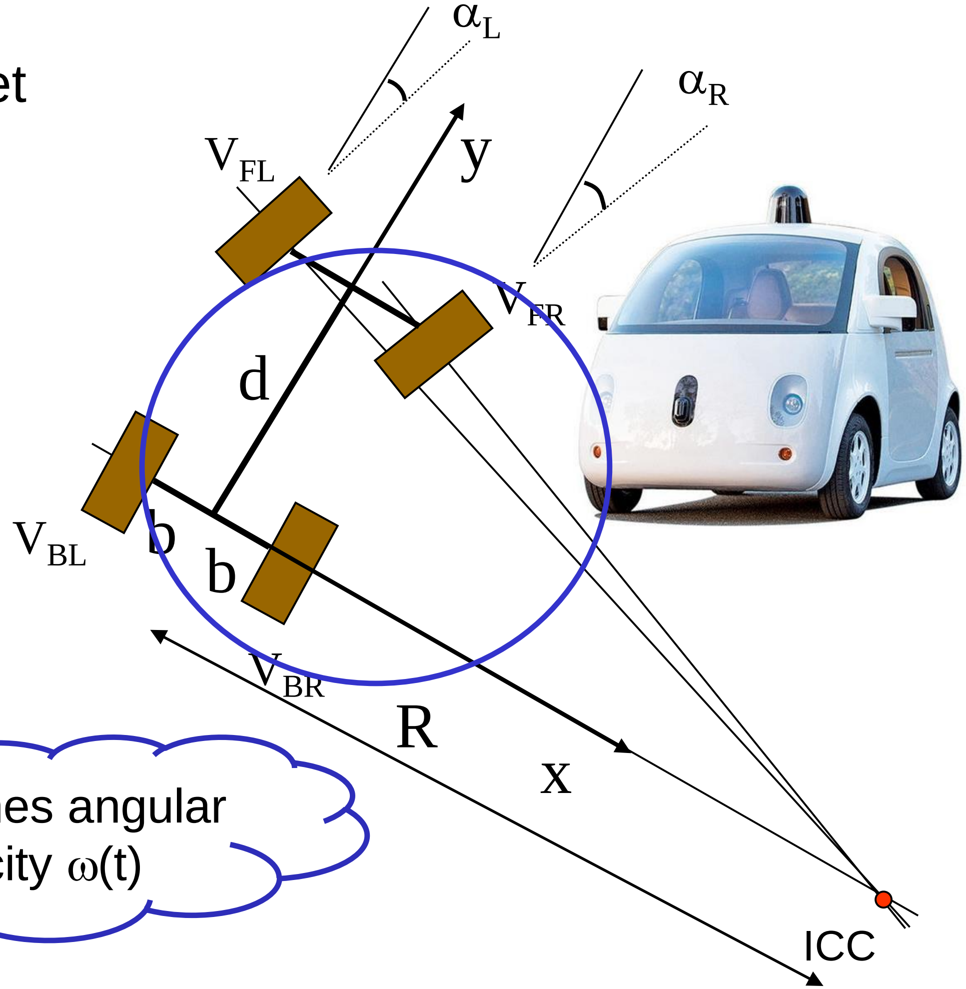
- Four wheels steering
- Wheels have limited turning angles
- No in-place rotation

Similar to the Trycicle model

$$R = \frac{d}{\tan \alpha_R} + b$$

$$\frac{\omega d}{\sin \alpha_R} = V_{FR}$$

Determines angular velocity ω(t)




Derive the rest as:

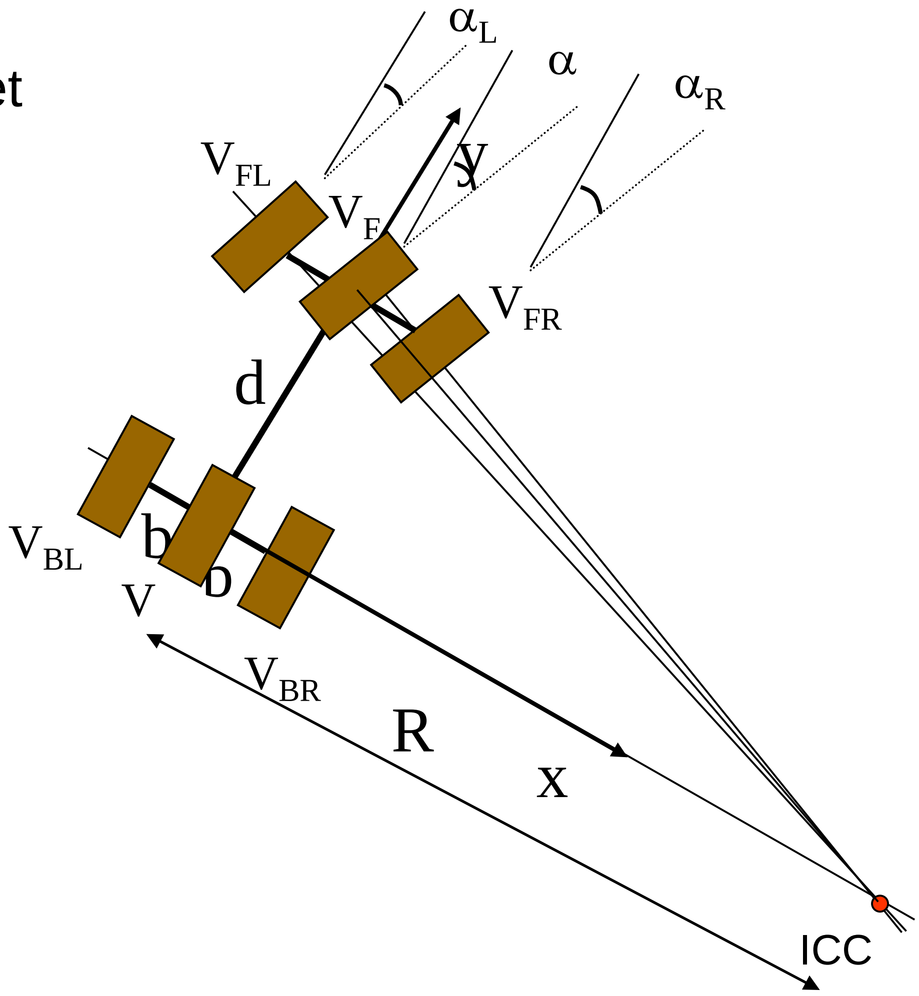
$$\frac{\omega d}{\sin \alpha_L} = V_{FL} \qquad \alpha_L = \tan^{-1}\left(\frac{d}{R+b}\right) \qquad \omega(R+b) = V_{BL} \qquad \omega(R-b) = V_{BR}$$

# Ackerman steering (bycicle approximation)



Most diffused kinematics on the planet

- Four wheels steering
- Wheels have limited turning angles
- No in-place rotation




Bycicle approximation

$$R = \frac{d}{\tan \alpha}$$

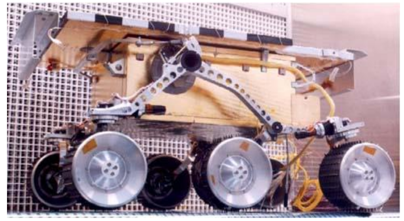
$$\frac{\omega d}{\sin \alpha} = V_F$$

Referred to the center of real wheels

$$\omega R = V \quad \Rightarrow \quad \omega = V \cdot \frac{\tan \alpha}{d}$$

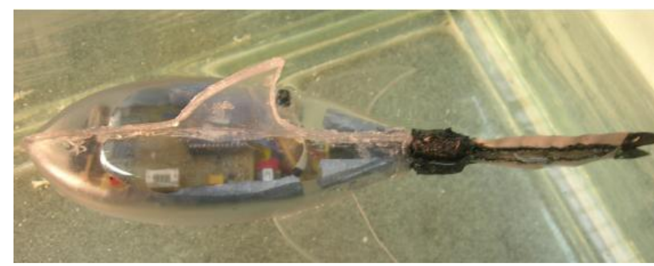
To move on the ground

- Multiple wheels
- Whegs
- Legs




To move in water

- Torpedo-like (single propeller)
- Bodies with thrusters
- Bioinspired



To move in air

- Fixed wings vehicles
- Mobile wings vehicles
- Multi-rotors